Circuit Equations

Choose as the dynamical variables:

- V_1: the voltage across capacitor C_1 (and the nonlinear resistance)
- V_2: the voltage across capacitor C_2 (and the voltage across the inductor)
- I: the current through the inductor.

Kirchoff's laws then give

$$\begin{aligned} C_1\frac{dV_1}{dt} &= R^{-1}(V_2 - V_1) - g(V_1) \\ C_2\frac{dV_2}{dt} &= -R^{-1}(V_2 - V_1) + I \\ L\frac{dI}{dt} &= -rI - V_2 \end{aligned}$$

where $g(V)$ is the nonlinear current-voltage characteristic for the effective nonlinear resistor (and is a negative quantity for the circuit). If we scale resistances by R_1, times by C_1R_1, measure voltages with respect to the switch point V_c in $g(V)$, and currents with respect to V_c/R_1, we get the equations

$$\begin{aligned} \frac{dX}{dt} &= a(Y - X) - \bar{g}(X) \\ \frac{dY}{dt} &= \sigma[-a(Y - X) + Z] \\ \frac{dZ}{dt} &= -c(Y + \bar{r}Z) \end{aligned}$$

with $X = V_1/V_c$, $Y = V_2/V_c$, $Z = R_1I/V_c$, and then the parameters of the equations are:

a	R_1/R	0.923
b	$1 - R_1/R_2$	0.636
c	$C_1R_1^2/L$	0.779
σ	C_1/C_2	0.066
$\bar{r}$	r/R_1	0.071

with the third column giving the values for the initial parameters of the applet. The nonlinear conductance is

$$\bar{g}(X) = \begin{cases} -X & |X| < 1 \\ [-1 + b(|X| - 1)]\, sgn(X) & 1 < |X| < 10 \\ [10(|X| - 10) + (9b - 1)]\, sgn(X) & |X| > 10 \end{cases}$$

where the expression for $|X| > 10$ is needed for stability, and corresponds to complicated saturation effects in the actual circuit. Note that the slope is -1 for $|X| < 1$, $-b$ for $1 < |X| < 10$, and 10 for $|X| > 10$.

The time independent solutions are at

$$X = \pm \frac{1-b}{\frac{a}{1+\bar{r}a} - b} \simeq \pm \frac{1-b}{a-b}, \quad Y = \frac{a\bar{r}}{1+\bar{r}a} X \simeq 0, \quad Z = -\frac{a}{1+\bar{r}a} X \simeq -aX \quad .$$

Linearizing about the fixed points gives solutions varying as $e^{\lambda t}$ with λ given by the eigenvalues of the stability matrix

$$\begin{bmatrix} -a+b & a & 0 \\ \sigma a & -\sigma a & \sigma \\ 0 & -c & -\bar{r}c \end{bmatrix}$$

and positive λ means the stationary solutions are unstable.

Some examples of the eigenvalues λ_1, λ_2, and λ_3:

a	b	c	σ	$\bar{r}$	λ_1	$\lambda_{2,3}$
0.923	0.636	0.779	0.066	0.071	$-.40191$	$-6.5991 \times 10^{-4} \pm .17715i$
1	0.636	0.779	0.066	0.071	$-.48631$	$5.0174 \times 10^{-4} \pm .1836i$
0.923	0.636	0.779	0.066	0	$-.40617$	$2.9125 \times 10^{-2} \pm .18836i$
1	0.636	0.779	0.066	0	$-.48897$	$2.9486 \times 10^{-2} \pm .1934i$

In each case there is one decaying (negative) eigenvalue, and a pair of oscillating (complex) eigenvalues, with an imaginary part around 0.2, corresponding roughly to the $1/\sqrt{LC_2}$ oscillation frequency, and a real part that is either slightly negative (decaying oscillation) as for the paraemters of the applet (first row) or slightly positive (growing oscillation) for the other rows.