

Physics 127c: Statistical Mechanics

Vortex Lines

Topological defects play a fundamental role in the properties of broken symmetry systems. In solids, for example, dislocations limit the maximum strain or stress that the solid can support, and initiate plastic flow beyond the elastic limit. In a superfluid or superconductor vortex lines play this role. Usually it is the physics of vortex lines that limit the flow velocity, rather than the Landau critical velocity derived from the excitation spectrum.

Topological Nature of Vortex Lines

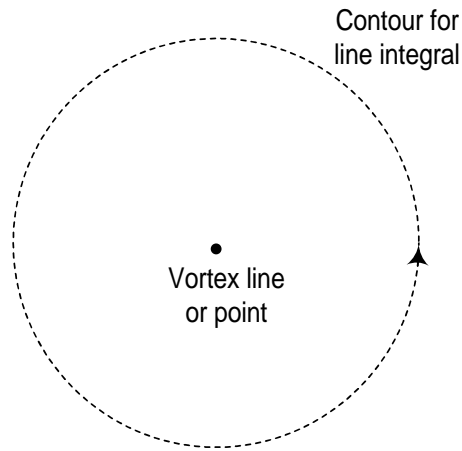


Figure 1: Contour integral defining quantized vortex line.

Since $e^{i\Phi}$ is single valued, we can imagine a configuration of the superfluid order parameter in which the phase increases by an integer times 2π as we circle a line in bulk 3d fluid or a point in thin film 2d fluid, Fig. 1

$$\oint \nabla \Phi \cdot \mathbf{dl} = n \times 2\pi \quad (1)$$

with n a positive or negative integer. We can smoothly distort the contour, and the (quantized) value of the integral will not change. There must be some point within the contour at which the phase is not defined. In the simplest case there will be only one such point, and this is the quantized vortex of strength n . Of course, it is only the phase description that has a singularity: the magnitude of the order parameter will smoothly go to zero at this point (line in 3d), over a microscopic length scale (the correlation length for variations of the magnitude).

Since we learn about the existence of a particular “defect” (i.e. higher energy configuration) of the system with nontrivial short scale structure purely from the behavior at large distances, such a defect is called topological. A topological configuration is robust, because small changes cannot alter the behavior of the order parameter far away to change the quantized integral. The elementary defects $n = \pm 1$ can only be prepared in pairs, not individually. The full topological structure is given by what is known as homotopy theory. This is the group theory of the mapping of the space of degenerate order parameters (here the circle of possible phase angles) onto the space—line, surface, points—at large distances containing and defining

the defect (here the encircling contour). There is a nice review article on homotopy theory by David Mermin (see Further Reading, below).

Energetics of Quantized Vortex Lines

It is easiest to first consider the two dimensional case where the vortex is a point defect. The energy (free energy at nonzero T) is given by

$$E = \frac{1}{2} \bar{\rho}_s \int (\nabla \Phi)^2 d^2x = \frac{1}{2} \rho_s \int v_s^2 d^2x. \quad (2)$$

To save writing lots of factors of \hbar/m sometimes it is useful to use $\bar{\rho}_s$ related to the usual ρ_s by $\bar{\rho}_s = (\hbar/m)^2 \rho_s$.

For a single quantized vortex of strength n located at the coordinate origin the phase field is $\Phi = n\phi$ with ϕ the polar angle, so that

$$\mathbf{v}_s = n \frac{\hbar}{m} \frac{1}{r} \hat{\phi} \quad (3)$$

($\hat{\phi}$ is a unit vector in the azimuthal direction), and the energy is

$$E_v = n^2 \times \frac{1}{2} \bar{\rho}_s \int dr 2\pi r r^{-2}. \quad (4)$$

The integral as written diverges at both small and large r . At small r we must recognize that the phase description breaks down, so that we must cut the integral off at some lower cutoff, or core size, a , and include a core contribution from the region where the magnitude of the order parameter decreases to zero. Then if we suppose that the system is a disc of radius R we get

$$E_v = n^2 \times \pi \bar{\rho}_s \ln \left(\frac{\alpha R}{a} \right), \quad (5)$$

where the number α gives the contribution from the core region. In three dimensions, for a straight vortex line we would simply multiply this result by L the length of the line. Since the energy increases as n^2 , so that an $n = 2$ line would be expected to split into two $n = 1$ lines, from now on we will only consider the $n = \pm 1$ lines.

Since the energy is infinite as $R \rightarrow \infty$ we might expect these configurations to never be important, but this is not the case, particular in two dimensions. We begin to see this by considering the energy of a pair of \pm vortices at separation s . To consider more complicated geometries such as this it is convenient to play some tricks on the energy integral. Note that we can write

$$E = \frac{1}{2} \bar{\rho}_s \int \nabla \cdot (\Phi \nabla \Phi) - \Phi \nabla^2 \Phi d^2x. \quad (6)$$

The second term is zero, since $\nabla^2 \Phi = 0$ is the Euler-Lagrange equation for minimizing the energy with respect to the configuration $\Phi(\mathbf{x})$. The first term can be reduced to a surface integral by the divergence theorem. However we must be careful since Φ is not single valued to include one or more mathematical cuts across which the phase may jump by $\pm 2\pi$. Usually the contribution from the bounding contour at infinity will disappear, so that only the contribution from these cuts remain

$$E = \pi \bar{\rho}_s \int_{\text{cuts}} \nabla \Phi \cdot d\Sigma \quad (7)$$

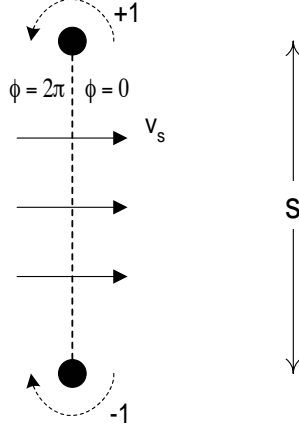


Figure 2: Calculation of energy of pair of vortices

where the integral is a “surface integral” over the cut (a line in 2 dimensions) and $d\Sigma$ is in the direction normal to the cut in the direction of the 0 to 2π jump. It is easy to see this result reproduces Eq. (6) (for $|n| > 1$ we would have $|n|$ cuts).

The energy of a pair of ± 1 vortices with separation s is now easy to calculate. We can place the cut between the vortices, and then calculate $\nabla\Phi$ here by superimposing the two vortex solutions—one reason that Eq. (7) is easier to use than the original expression is that it is linear in the velocity, so we may use superposition. This gives

$$E_{\text{pair}} = 2\pi \bar{\rho}_s \ln\left(\frac{\alpha s}{a}\right) = 2\pi \rho_s \left(\frac{\hbar}{m}\right)^2 \ln\left(\frac{\alpha s}{a}\right). \quad (8)$$

A vortex pair acts like a smoke ring: the velocity field from one vortex carries the other along, so that the pair propagates at a speed

$$v_{\text{pair}} = \frac{\hbar}{m} \frac{1}{s} \quad (9)$$

normal to the line joining the pair. We can also define an effective momentum (also known as the impulse)

$$p_{\text{pair}} = 2\pi \rho_s (\hbar/m) s. \quad (10)$$

This is then consistent with the Hamiltonian expression $v_{\text{pair}} = dE/dp_{\text{pair}}$. The energy E_{pair} is finite for finite s . This means that in two dimensions vortex pairs may be excited thermally, and this turns out to be key to understanding the phase transition.

In three dimensions a pair of vortex lines have an energy proportional to the length. Vortex rings have a finite energy, of order $\bar{\rho}_s r \ln(\alpha r/a)$ for rings of radius r , and so again may be excited thermally. These thermally excited rings are important in limiting the superflow velocity, but do not in general provide a way to understand the phase transition.

Breakdown of Superflow

Vortex pairs in 2d or lines or rings in 3d provide a mechanism for limiting the superflow velocity. Again we will first consider the 2d case.

Consider flow around a 2d ring corresponding to a circulation of M quanta (i.e. $\oint \nabla\Phi \cdot d\mathbf{l} = 2M\pi$). If we “unwrap” the ring, we have flow in a rectangle of length L with periodic boundary conditions in one

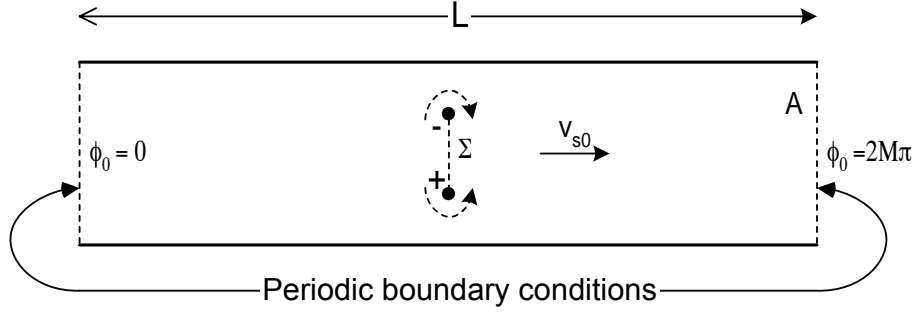


Figure 3: Geometry for considering the breakdown of superflow due to a vortex pair.

direction, and the superfluid speed is $v_{s0} = (\hbar/m)2\pi M/L$, Fig. 3. Consider a vortex pair of separation s oriented as shown. The flow from the vortex pair tends to cancel v_{s0} . In fact if the pair grows from infinitesimal separation to span the channel, the circulation is reduced by one quantum: a vortex pair provides a dissipation mechanism for superflow! Lets calculate the energetics of this process.

We can use Eq. (7) now with two cuts: one at the vortex pair, and one (of strength M) at the ends. The energy is

$$E = E_0 + E_{\text{pair}}(s) - \pi\rho_s(\hbar/m) \int \mathbf{v}_{s0} \cdot \mathbf{d}\Sigma - \pi\rho_s(\hbar/m)M \int \mathbf{v}_{s,\text{pair}} \cdot \mathbf{d}\mathbf{A} \quad (11)$$

where $E_0 = \frac{1}{2}\rho_s v_{s0}^2 \Omega$ is the flow energy without the vortex pair with Ω the area, E_{pair} is the pair energy in the absence of the flow, and the last two terms give the interaction energy between the pair and the flow v_{s0} with the first term the integral of the flow \mathbf{v}_{s0} over the cut between the pair, and the last term the integral of the superfluid velocity induced by the ring over the end section of the channel. The last two terms are in fact equal¹. This gives

$$E = E_0 + 2\pi(\hbar/m)^2\rho_s \left[\ln\left(\frac{\alpha s}{a}\right) - \frac{v_{s0}}{\hbar/m}s \right]. \quad (12)$$

You can also calculate the last term as a $\mathbf{p}_{\text{pair}} \cdot \mathbf{v}_{s0}$ term with \mathbf{p}_{pair} given by Eq. (10). As a function of s this expression is positive for small s (actually the \ln suggests a negative energy for small enough s , but this is unphysical) increasing to a maximum at a separation $s_{\text{max}} = \hbar/mv_{s0}$, and then decreasing and becoming negative for large enough s , when the energy gained by the interaction with v_{s0} dominates. The maximum energy is

$$\Delta E_{\text{max}} = 2\pi(\hbar/m)^2\rho_s \ln\left(\frac{\gamma\hbar}{mv_{s0}}\right) \quad (13)$$

with γ some numerical constant.

We can imagine two mechanisms for the breakdown of superflow.

For the first, we might imagine an “extrinsic mechanism” where geometrical imperfections on a scale of d might provide nucleation sites for vortex pairs for velocities $v_{s0} > v_c = \hbar/md$. The estimate of v_c depends on the scale of the imperfections, but is typically less than the Landau critical velocity.

For $k_B T \gtrsim \Delta E_{\text{max}}$ thermal fluctuations over the energy barrier may occur and provide an “intrinsic mechanism” for the decay. In fact there is always *some* probability of such excitations proportional to the Boltzmann factor

$$e^{-\Delta E_{\text{max}}/k_B T} \propto v_{s0}^{2\pi(\hbar/m)^2\rho_s/k_B T}. \quad (14)$$

¹This can be seen by a simple magnetic analogy. The circulation plays the role of a current, and \mathbf{v}_s the magnetic field. The two expressions are then different ways of calculating the mutual inductance between the pair and the current sources of v_{s0} which is like a solenoid.

This gives a decay rate proportional to a *power* of the superfluid velocity

$$\dot{v}_s = -\gamma v_s^p. \quad (15)$$

We will see in fact that the transition temperature into the superfluid state is given by

$$(\pi/2)(\hbar/m)^2 \rho_s(T_c) = k_B T_c, \quad (16)$$

so that near T_c the power is $p = 4$. There is strictly dissipation for all flow velocities, and *no* critical velocity in an ideal system in two dimensions. In practice the vortices may be pinned by substrate imperfections, and persistent currents may then result.

In three dimensions vortex rings play the same sort of role as vortex pairs in two dimensions. The more complicated geometry makes the calculations a little harder. You can find the details in the paper by Langer and Fisher (see Further Reading). The expressions corresponding to Eqs. (5)-(10) for a ring of radius r are

$$E_{\text{ring}} = \pi \bar{\rho}_s 2\pi r \ln\left(\frac{\bar{\alpha}r}{a}\right), \quad (17)$$

$$v_{\text{ring}} = \frac{\hbar}{m} \frac{1}{2r} \ln\left(\frac{\bar{\gamma}r}{a}\right), \quad (18)$$

$$p_{\text{ring}} = 2\pi \rho_s (\hbar/m) \pi r^2, \quad (19)$$

with $\bar{\alpha}$, $\bar{\gamma}$ numbers of order unity. The energy $E_{\text{ring}} - p_{\text{ring}} v_{s0}$ has a maximum of order

$$\Delta E_{\text{max}} \sim (\pi^2/2) \rho_s (\hbar/m)^3 v_{s0}^{-1} \quad (20)$$

at a radius

$$r_{\text{max}} \sim \frac{\hbar}{m} \frac{1}{2v_{s0}} \quad (21)$$

(ignoring logarithmic factors). This gives the same estimate of extrinsic critical velocity. Now however, the thermal activation process involves the Boltzmann factor

$$e^{-\Delta E_{\text{max}}/k_B T} \sim \exp\left[-A \frac{\pi^2}{2} \left(\frac{\hbar}{m}\right)^3 \frac{\rho_s}{k_B T v_{s0}}\right] \quad (22)$$

where A takes care of the logarithmic factors. Except near T_c where ρ_s goes to zero this is a negligible rate, and the extrinsic mechanisms dominate. However near T_c this predicts a critical velocity (such a function “turns on” so rapidly that we may still define a critical velocity)

$$v_c \sim A \frac{\pi^2}{2} \left(\frac{\hbar}{m}\right)^3 \frac{\rho_s}{k_B T} \propto \left(1 - \frac{T}{T_c}\right)^{2/3} \quad (23)$$

using $2/3$ as the approximate value of the superfluid density exponent. This temperature dependence is indeed seen in experiment, first by Clow and Reppy.

As a final link in the discussion of flow dissipation by vortices, we can use the dynamic Josephson relation to evaluate the pressure (or alternatively the temperature difference) needed to drive the flow through a tube. Consider again Fig. 3, but now imagine this is a flow channel with a pressure across the ends, rather than with periodic boundary conditions. The dynamic Josephson equation is

$$\hbar \Delta \dot{\Phi} = -\Delta\mu = -(\Delta P - s \Delta T)/n, \quad (24)$$

(with n the number of particles per volume and s the entropy per volume). The rate of change of the phase difference between the ends of the tubes is just 2π times the number of vortex pairs or rings that expand across the channel per unit time, which is just the production rate in steady state. (Alternatively, single vortex lines could grow out of one edge, e.g. nucleated by imperfections, and propagate across the channel.) Hence a steady rate of vortex production must be balanced by a driving pressure or temperature difference.

Further Reading

The review article by Anderson referenced in [Lecture 4](#) remains useful. The article *Topological Theory of Defects* by *David Mermin*, *Rev. Mod. Phys.* **51**, 591 (1979) ([online](#)) gives a thorough discussion of topological defects and homotopy theory. Early references on the critical velocity of superflow due to thermal activation of vortex rings are *Clow and Reppy*, *Phys. Rev. Lett.* **19**, 291 (1967), and *Langer and Fisher*, *Phys. Rev. Lett.* **19**, 560 (1967).

Revised December 23, 2013