Physics 127c: Statistical Mechanics

Statistical Mechanics of Superfluidity

Excitation Picture

For the description in terms of a flowing ground state plus excitations see Lecture 15 and problem 3 of Homework 7 for Ph127a.

This approach has some disadvantages. Conceptually, it has the problem that we calculate superflow in terms of what it isn't, i.e. the excitations which reduce the mass flow! The superflow is slipped in by Galilean boost arguments. Practically, the approach is limited to translationally invariant situations, such as a bulk fluid, where Galilean invariance applies. The most dramatic manifestations of superfluidity—the flow through porous material with pores of a few Å in size such as vycor or aerogel, or in thin films of a few atomic layers thick (or even subatomic layers) on substrates, cannot be addressed.

Broken Symmetry Approach

A more profound approach is to understand superfluidity within the general picture of broken symmetries. The "Bose condensation" assumption $\langle b_0 \rangle \neq 0$ (and is a macroscopic quantity) is a statement of the broken symmetry. Rather than using the momentum representation, it is more convenient for further development to rewrite this in terms of the boson field operator: in the lowest energy state (lowest free energy at nonzero temperature)

$$\langle \psi(\mathbf{x}) \rangle = \psi_0 = |\psi_0| \, e^{i\phi}. \tag{1}$$

Here $|\psi_0|$ gives the *strength* of the ordering temperature, is given by the condensate density $|\psi_0| = \sqrt{n_0}$, so it is fixed by the thermodynamic parameters P, T etc. On the other hand ϕ can take on any value, and the particular value in the superfluid reflects the broken symmetry, just as the direction of the magnetization in a ferromagnet. The phase ϕ is actually the phase of the state into which Bose condensation occurs in the weakly interacting limit when this description makes sense. The broken symmetry is called *phase symmetry* (the invariance of the Hamiltonian to a change of phase of the quantum wavefunction) or *gauge symmetry*, since for a charged system the wave function phase changes if the gauge of the electromagnetic vector potential is changed.

Since an overall phase change does not change the energy, there are low energy states with a slow spatial variation of the phase $\phi(\mathbf{x})$, or with a phase difference $\Delta\phi$ between two weakly coupled superfluids (the coupling in the latter case is called a *Josephson junction*). These situations lead to a flow of particles. Since these are equilibrium situations, the flow is dissipation free, i.e. superflows.

Superflow from phase differences/gradients

Josephson junctions Consider two weakly coupled lumps of superfluid (e.g. weakly interacting Bose gases). We can imagine the weak coupling as resulting from a point contact: early Josephson junction in superconductors were indeed made simply by bringing a superconducting spike into contact with a second piece of superconductor. We can model this situation by the Hamiltonian

$$H = H_1 + H_2 - T[\psi_1^+(\mathbf{0})\psi_2(\mathbf{0}) + \psi_2^+(\mathbf{0})\psi_1(\mathbf{0})], \tag{2}$$

where $H_{1,2}$ are the Hamiltonians of the separate systems, and the last term acts to transfer particles between 1 and 2 at the contact point $\mathbf{0}$ (this term is known as the tunnelling Hamiltonian in the context of superconductors).

To calculate the energy to lowest order in the coupling strength T we can simply replace the field operators by their expectation values $\sqrt{n_0}e^{i\phi_{1,2}}$ with ϕ_1, ϕ_2 the phases of the two superfluids, to give

$$E = E_0 - 2Tn_0\cos(\phi_1 - \phi_2), \tag{3}$$

with E_0 the energy of the uncoupled systems. Note that the energy is minimized for $\phi_1 = \phi_2$ for the sign of interaction term I chose—the usual case.

We can also calculate the particle transfer between the two systems. In the Heisenberg picture

$$\frac{dN_1}{dt} = \frac{i}{\hbar}[H, N_1], \tag{4}$$

with $N_1 = \int_1 d\Omega\psi^+(\mathbf{x})\psi(\mathbf{x})$ the number of particles in 1. H_1 and H_2 commute with N_1, so the only time dependence comes from the tunnelling term. Using the commutation rules for the field operators gives

$$[\psi(\mathbf{0}), N_1] = \psi(\mathbf{0}), \tag{5}$$

$$\left[\psi^+(\mathbf{0}), N_1\right] = -\psi^+(\mathbf{0}), \tag{6}$$

so that

$$\frac{dN_1}{dt} = \frac{i}{\hbar}T\left(\psi_1^+(\mathbf{0})\psi_2(\mathbf{0}) - \psi_2^+(\mathbf{0})\psi_1(\mathbf{0})\right). \tag{7}$$

Taking the expectation value in the ground state, and using Eq. (1) gives the particle flow

$$J_{2\to 1} = \frac{dN_1}{dt} = \frac{2Tn_0}{\hbar}\sin(\phi_1 - \phi_2). \tag{8}$$

Equations (3) and (8) were derived by Josephson, and are named after him. The second equation shows that a phase difference leads to a flow of particles. Since this is an equilibrium configuration, this flow occurs without dissipation.

Josephson also derived a dynamical equation for how the phase evolves. For a single system we have, taking the expectation value of the Heisenberg equation of motion in the ground state

$$\left\langle 0\left|\frac{d\psi}{dt}\right|0\right\rangle = \frac{i}{\hbar}\langle 0\,|[H,\psi]|\,0\rangle = \frac{i}{\hbar}\langle 0\,|H\psi - \psi H|\,0\rangle. \tag{9}$$

Since ψ adds a particle the H in the first term acts on the component of the ground state with N particles and gives $E_{0,N}$ and the one in the second term acts on the component with $N+1$ particles and gives $E_{0,N+1}$. The difference is just the chemical potential

$$E_{0,N+1} - E_{0,N} = \mu, \tag{10}$$

and so

$$\frac{d\psi_0}{dt} = -\frac{i}{\hbar}\mu\psi_0. \tag{11}$$

The solution is $\psi_0 \propto e^{-\iota\mu t/\hbar}$, or an equation for the dynamics of the phase

$$\hbar\frac{d\phi}{dt} = -\mu. \tag{12}$$

For the two superfluids in contact

$$\hbar\frac{d(\phi_1 - \phi_2)}{dt} = -(\mu_1 - \mu_2). \tag{13}$$

Equations (3), (8) and (12) were derived by Josephson, and are named after him. The second equation shows that a phase difference leads to a flow of particles. Since this is an equilibrium configuration, this flow occurs without dissipation.

Schematically, the equations can be summarized by

$$\hbar \frac{dN}{dt} = \frac{dE}{d\phi}, \tag{14a}$$

$$\hbar \frac{d\phi}{dt} = -\frac{dE}{dN}. \tag{14b}$$

These equations can be interpreted in terms of ϕ and $N\hbar$ being conjugate variables analogous to x and p, so that the quantum mechanical behavior can be calculated using the commutation rule $[\phi, N] = -i$ or $N \to -i d/d\phi$. We are not very familiar in standard quantum mechanics dealing with number and phase operators. A simple "lattice-gas model" of a superfluid you will look at in Homework 3 shows the analogy with a magnetic system with the correspondence $N \leftrightarrow S_z$ (the z-component of the total spin) and ϕ corresponding to rotations about the z axis, so that the conjugate-variable description is apparent.

Slow phase gradients Within a superfluid system slow spatial variations of the phase are low energy states. Writing

$$\mathbf{v}_s = (\hbar/m)\nabla\phi \tag{15}$$

there will be an energy cost per unit volume

$$\varepsilon = \frac{1}{2}\rho_s v_s^2. \tag{16}$$

By similar arguments to the previous section, or more macroscopic ones, the phase gradient leads to a mass flow or momentum density

$$\mathbf{g} = \rho_s \mathbf{v}_s. \tag{17}$$

I have chosen to write things in terms of $\mathbf{v}_s$ which has the dimensions of a velocity. For a translationally invariant system where we can use Galilean boost arguments, $\mathbf{v}_s$ is indeed a velocity, and m is the mass of the particles. In other situations the idea of a velocity is not useful, the definition is just a convention, and the mass used could be anything (of course, with the corresponding change of the value of ρ_s).

Note that the superfluid density ρ_s is introduced in terms of the energy cost of spatial variations. It is fundamentally the *stiffness constant* of the broken symmetry variable, entirely analogous to the elastic constants of a solid, of the gradient coefficient in a Ginzburg-Landau free energy. It is then a result of the fundamental connection between the number and phase operators discussed below Eq. (14) that leads to its role as the coefficient relating the phase gradient to a mass flow Eq. (17).

Equation (15) tells us many important things about superflow. Clearly it is "potential flow"

$$\nabla \times \mathbf{v}_s = 0. \tag{18}$$

Also, since the potential ϕ corresponds to a quantum phase, so that $e^{\iota\phi}$ must be single valued, the circulation (the integral around a closed loop) is quantized

$$\oint \mathbf{v}_s \cdot \mathbf{dl} = n \times 2\pi\,\hbar/m, \qquad n \text{ and integer.} \tag{19}$$

The quantity $\kappa = h/m$ is the quantum of circulation: in a closed torus geometry only flows corresponding to circulations equal to integer multiples of this basic unit are possible. In a "bucket" rotating with angular velocity Ω the fluid velocity cannot simply be the obvious $\Omega \times \mathbf{x}$, since this violates the quantized circulation. This leads us to the topic of *quantized vortex lines* which are the topological defects of the broken symmetry state, and play an important role in superflow properties in general, and the nature of the phase transition in two dimensions.

Further Reading

Feynman has a typically elegant discussion of the role of the phase in superfluidity, superconductivity, and the Josephson effect in chapter 21 of volume III of his *Lectures on Physics*. A classic review article from the early days of the understanding of broken symmetry is Rev. Mod. Phys. **38**, 298 (1966) by *P. W. Anderson*, available online. The visualization of the vortex lines in rotating He^4 was done by *E. J. Yarmchuk, M. J. V. Gordon, and R. E. Packard*, Phys. Rev. Lett. **43**, 214 (1979), available online. *Brian Josephson*'s original paper is Phys. Lett. **1,** 251 (1962), and he wrote a review at Rev. Mod. Phys. 36, 216 (1964), available online. *David Goodstein* has a good discussion in *States of Matter*.