Physics 127c: Statistical Mechanics

Feynman Diagrams

Using the language of second quantization it is now possible to develop the perturbation theory in the interaction. As in the classical case a diagrammatic formulation is found to be convenient.

The first issue is what do we calculate? We want to look at something simple enough that it is something we care about, usually some low order correlation function involving measurements of a few particles, but rich enough to contain useful results. Although direct expansions of the ground state energy, or the free energy at finite temperature, are done, formulating the expansion in terms of the "propagator", is often preferred. This gives us *Feynman diagrams* that have close analogies with the diagrams of quantum electrodynamics. The following is meant only to be a *very* brief introduction, and I will only state a few results, without any derivation.

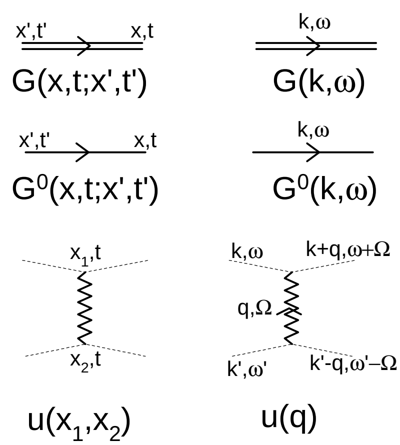

Figure 1: Dictionary for Feynaman diagrams. G is the propagator of the interacting system; G^0 that of the noninteracting system; and u is the pairwise interaction (the dotted lines show where the propagators are to be attached).

The propagator is defined as

$$G_{\alpha\beta}(\vec{x},t;\vec{x}',t') = \begin{cases} -i\left\langle 0\left|\psi_\alpha(\vec{x},t)\psi_\beta^+(\vec{x}',t')\right|0\right\rangle & \text{for } t > t' \\ \mp i\left\langle 0\left|\psi_\beta^+(\vec{x}',t')\psi_\alpha(\vec{x},t)\right|0\right\rangle & \text{for } t < t' \end{cases} \tag{1}$$

with the upper sign for Bosons and the lower sign for Fermions in the second expression. Here $|0\rangle$ is the ground state of the interacting system. Note this is different notation than in the last lecture, where I used this symbol for the no-particle state—you can however think of the ground state as the "no excitation" state, so there is some logical connection. (The notation G is because the propagator is a Green function in an equation for the dynamics, but this is not a very useful connection.) The propagator answers the following question for $t > t'$: at time t'add a particle in spin state β at the position $\vec{x}'$to the ground state, and then at a later time t remove a particle in spin state α at position $\vec{x}$ and ask for the overlap of the state produced with the ground state. For $t < t'$ we first remove a particle, and then add it back at the later time, i.e. we look at the propagation of a "hole".

The Fourier transform of this propagator (in a translationally invariant, time independent system where the dependence is just on $\vec{x} - \vec{x}'$ and $t - t'$) is

$$G_{\alpha\beta}(\vec{k}, t - t') = \begin{cases} -i \left\langle 0 \left| c_{\vec{k}\alpha}(t) c^{+}_{\vec{k}\beta}(t') \right| 0 \right\rangle & \text{for } t > t' \\ \mp i \left\langle 0 \left| c^{+}_{\vec{k}\beta}(t') c_{\vec{k}\alpha}(t) \right| 0 \right\rangle & \text{for } t < t' \end{cases} \tag{2}$$

(the result would be zero if we had different $\vec{k}$ for the creation and annihilation operators, by momentum conservation). We can also Fourier transform with respect to $t - t'$ and get $G_{\alpha\beta}(\vec{k}, \omega)$. If there is no magnetic field or spontaneous magnetization $G_{\alpha\beta} = G\delta_{\alpha\beta}$—and this is the situation I will describe. The diagrammatic dictionary is shown in Fig. (1). Note that the potential is instantaneous (i.e. $u(x_1, x_2)$ is the interaction between particle 1 at x_1 at time t and particle 2 at x_2 at the same time t). In the Fourier representation this means the interaction is $u(\vec{q})$ independent of the frequency Ω.

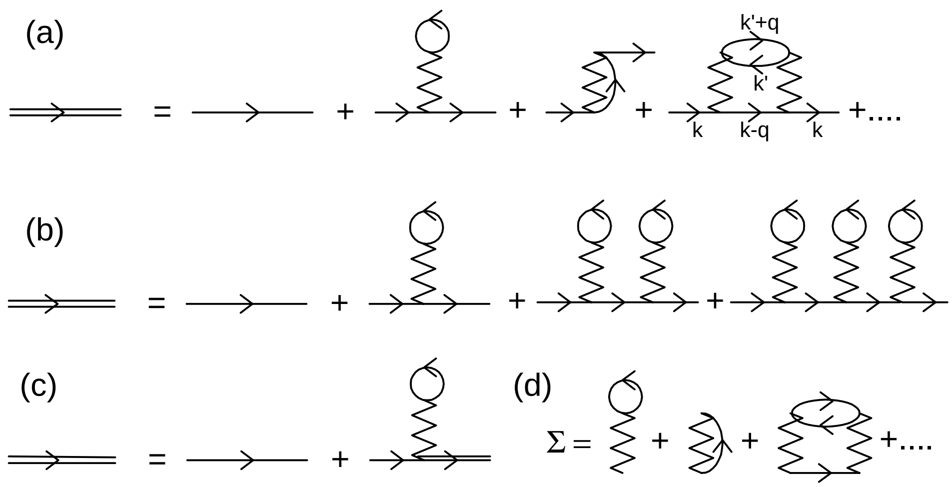

Figure 2: (a) The first few diagrams in the Fermion perturbation expansion. (b) A selected set of diagrams to infinite order. (c) The sum of (b) can be done diagrammatically: $G = G^0 + G^0 \Sigma_H G$ with Σ_H the "potential + closed fermion loop" diagram. (d) Expansion for the self energy $\Sigma = \Sigma_H + \cdots$. The labels are to be read as "4-vectors", i.e. $k \equiv \vec{k}, \omega$.

The diagrammatic perturbation expansion for the interacting propagator G in terms of the noninteracting propagator G_0 in powers of the interaction strength u for the Fermion case is shown in Fig. (2). (The Boson case is a little more complicated because of Bose condensation.) Think of time increasing from left to right, and backwards propagating lines are interpreted as holes. The up-down direction has no significance. There is one zeroth order diagram, two first order diagrams, and many second order diagrams of which only one is shown. These diagrams give an *intuitive* content to the effects of the interaction, and also have a *quantitative* interpretation. For example the last diagram in the figure can be described as follows: the added particle interacts with the ground state sea exciting a particle-hole pair, which then annihilates to give back the added particle and ground state. The first order diagrams are actually less intuitive: the first one corresponds to the interaction of the added particle with the mean density (known as the Hartree term) and the second to an "exchange" version of this with particle labels interchanged (known as the Foch term). The quantitative interpretation of the second order diagram is:

$$\left[\left(\frac{i}{\hbar} \right) \frac{1}{(2\pi)^2} \right]^2 (-1)^1 \, G_0(k) \int d^4k' \int d^4q \left[\tilde{u}(q) G_0(k - q) G_0(k') G_0(k' + q) \tilde{u}(q) \right] G_0(k) \tag{3}$$

where d^4k means $d^3k\, d\omega$. The translation from pictures to algebra should be reasonably apparent. As usual, the most difficult part is numerical prefactors: here there is a factor of $[(i/\hbar)(1/2\pi)^2]$ raised to the order (the number of u lines appearing), and a factor of (-1) raised to the power of the number of "closed Fermion loops".

As you might imagine, the noninteracting propagator is easy to calculate, since it corresponds to the free propagation of the added particle. The time dependence is just the phase propagation $e^{-iEt/\hbar}$ with E the energy of the state. The noninteracting ground state is the Fermi sea of filled states up to $k = k_F$. Since we can only then only add a particle for $k > k_F$ and take one away for $k < k_F$ the noninteracting propagator $G(\vec{k}, t - t')$ is nonzero for $t - t' > 0, k > k_F$ or $t - t' < 0, k < k_F$. Fourier transforming gives

$$G_0(\vec{k}, \omega) = \frac{\Theta(k - k_F)}{\omega - \varepsilon_k/\hbar + i\eta} + \frac{\Theta(k_F - k)}{\omega - \varepsilon_k/\hbar - i\eta} \tag{4}$$

where $\varepsilon_k = \hbar^2 k^2/2m$ is the noninteracting single particle energy, and η is a positive infinitesimal number that is needed to define the different behavior for $t - t' \gtrless 0$.

Just as in the cluster expansion, we often want to sum an infinite subclass of diagrams, for example the repeated interaction with the ground state shown in Fig. (2b). This infinite sum is equivalent to the diagrammatic statement in (c), as you can see by recursively replacing G on the right hand side with the full expression for G. Algebraically this corresponds to

$$G = G_0 + G_0 \Sigma_H G \tag{5}$$

or

$$G = \frac{1}{G_0^{-1} - \Sigma_H} = \frac{1}{\omega - \varepsilon_k/\hbar - \Sigma_H(\vec{k}, \omega)} \tag{6}$$

(the $i\eta$ pieces are hidden in Σ_H). Since the poles of G_0 in the complex ω plane were at the single particle energies, we might guess that the poles of G give the excitation energies (quasiparticle energies) of the interacting system, and so Σ_H is the first estimate of the interaction correction to the single particle energy. Higher order terms in Σ are shown in (d).

Further Reading

This has been a very brief introduction to Feynman diagrams, just to give you a teaser. We will not be using them again in the course. Some textbooks that focus on the application of these techniques to many body physics are *Quantum Theory Of Many-Particle Systems* by Fetter and Walecka, *A Guide To Feynman Diagrams In The Many-Body Problem* by Mattuck, and *Many-Particle Physics* by Mahan.