

Physics 127c: Statistical Mechanics

Fermi Liquid Theory: Thermodynamics

Energy Expansion

For a small number of excited quasiparticles the energy expanded about the ground state is

$$E = E_0 + \sum_{\mathbf{p},\sigma} \varepsilon_p \delta n_{\mathbf{p},\sigma} + O(\delta n^2) \quad (1)$$

where $\delta n_{\mathbf{p},\sigma}$ is plus one for every excited quasiparticle for $p > p_F$ and minus one for every quasihole (empty state in the noninteracting limit) for $p < p_F$. (I have switched to momentum rather than wave vector labels to be consistent with the nicest reference on this topic *Nozieres and Pines*.) Since the excitations have definite particle number, momentum and spin the total momentum and spin are

$$N - N_0 = \sum_{\mathbf{p},\sigma} \delta n_{\mathbf{p},\sigma}, \quad \mathbf{P} = \sum_{\mathbf{p},\sigma} \mathbf{p} \delta n_{\mathbf{p},\sigma}, \quad S_z = \sum_{\mathbf{p}} (\delta n_{\mathbf{p},\sigma\uparrow} - \delta n_{\mathbf{p},\sigma\downarrow}). \quad (2)$$

I have chosen to use a spin notation of “up” or “down” with respect to a convenient z axis rather than a more rotationally invariant formulation. The quantity ε_p is not the same as in the interacting system, but can be expanded in the same way about $p = p_F$

$$\varepsilon_p = \varepsilon_F + v_F(p - p_F) + \dots \quad (3)$$

where ε_F, v_F are not the same as in the noninteracting system. The single quasiparticle energy does not depend on spin in a spin invariant system. Since p_F is unchanged, an *effective mass* m^* is often defined as

$$v_F = p_F / m^*. \quad (4)$$

Using these expressions we can calculate the density of states of the single quasiparticle excitations. An energy band $\Delta\varepsilon$ corresponds to a momentum shell near p_F of width $\Delta\varepsilon/v_F$. In this band there are

$$\Delta N_\sigma = \frac{V}{(2\pi\hbar)^3} 4\pi p_F^2 \frac{\Delta\varepsilon}{v_F} \quad (5)$$

states of one spin type, with V the volume of the system. The density of states (per energy per volume) for each spin is written $N(0)$, where the 0 refers to “at zero energy relative to the Fermi surface”. Thus

$$N(0) = \frac{m^* p_F}{2\pi^2 \hbar^3}. \quad (6)$$

This is changed from the noninteracting gas by the factor m^*/m .

It is usually convenient to look at the free energy $F = E - \mu N$ (this is not the Helmholtz free energy $A = E - TS$), and subtracting off the ground state value F_0 since we are interested in the changes from the ground state due to the excitations

$$F - F_0 = \sum_{\mathbf{p},\sigma} (\varepsilon_p - \mu) \delta n_{\mathbf{p},\sigma} + \dots \quad (7)$$

Since at zero temperature $\mu = \varepsilon_F$, and $\varepsilon_p - \varepsilon_F$ is small for $p \simeq p_F$ where quasiparticles are well defined objects, this term is “second order small”. Landau had the significant insight to realize that for consistent answers the $O(\delta n^2)$ terms must also be retained, so that

$$F - F_0 \simeq \sum_{\mathbf{p},\sigma} (\varepsilon_p - \mu) \delta n_{\mathbf{p},\sigma} + \frac{1}{2V} \sum_{\substack{\mathbf{p},\mathbf{p}' \\ \sigma,\sigma'}} f(\mathbf{p},\sigma; \mathbf{p}',\sigma') \delta n_{\mathbf{p},\sigma} \delta n_{\mathbf{p}',\sigma'}. \quad (8)$$

Equation (8) introduces the *effective interaction* $f(\mathbf{p}, \sigma; \mathbf{p}'\sigma')$ between the quasiparticle. The prefactor $1/2V$ is chosen in comparison with the usual expression for the interaction between particles. We know very little about the effective interaction, except the properties deriving from symmetries, and that it should vary smoothly with its arguments. Since $p \simeq p_F$ for quasiparticles to be well defined, we can ignore the dependence on p, p' , and by rotational invariance this means f can only depend on $\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}'$ i.e. the angle between \mathbf{p} and \mathbf{p}' . By spin rotation symmetry there is just a spin-parallel, and spin-antiparallel interaction. This means we can write

$$f(\mathbf{p}, \sigma; \mathbf{p}'\sigma') = \sum_{l=0}^{\infty} \left(f_l^{(s)} + \sigma\sigma' f_l^{(a)} \right) P_l(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}') \quad (9)$$

where the angular dependence is expanded in Legendre polynomials, $\sigma\sigma' = 1$ for parallel spins and -1 for antiparallel spins, and the $f_l^{(s,a)}$ are *constants*, that depend of course on the nature of the physical system, pair potential etc. Since $f_l^{(s,a)}$ has the dimensions of energy times volume, it is conventional to introduce dimensionless *Fermi liquid parameters* $F_l^{(s,a)}$ through

$$F_l^{(s,a)} = 2N(0) f_l^{(s,a)} \quad (10)$$

with $2N(0)$ the total density of states per energy per volume at the Fermi surface.

The quasiparticle interaction is still parameterized in terms of a discrete infinity of constant that must be found by other means (experiment or microscopic theory) but in practice interesting physical properties often just depend on small l angular components. Fermi liquid theory then provides a phenomenological theory in terms of the small l interaction parameters $F_l^{(s,a)}$.

Some examples are given in the next few subsections. In all the calculations it should be remembered that only excitations for p near p_F can be considered, and we can usually approximate $\mathbf{p} \simeq p_F \hat{\mathbf{p}}$ with $\hat{\mathbf{p}}$ the unit vector in the direction of \mathbf{p} , so that the integration over the radial $|p|$ direction is usually trivial, and only the angular integration is left.

The calculations can often be formulated in terms of infinitesimal perturbations of the Fermi sea—indeed we will see in the next lecture that collective modes such as sound can be understood in terms of the “ringing” of the Fermi sea. For a displacement of the Fermi surface by δp_F at momentum $\hat{\mathbf{p}} p_F$ we then have

$$\delta n_{\mathbf{p},\sigma} = -\frac{dn_0}{d\varepsilon_p} v_F \delta p_F(\hat{\mathbf{p}}) \quad (11)$$

with $n_0(\varepsilon_p)$ the Fermi function. At zero temperature the Fermi function is a step function, and so

$$\delta n_{\mathbf{p},\sigma} = \delta(\varepsilon_{\mathbf{p}} - \varepsilon_F) v_F \delta p_F(\hat{\mathbf{p}}). \quad (12)$$

Since this is a singular function, I prefer to formulate the calculation directly in terms of the relevant sums, such as $\sum_{\mathbf{p},\sigma} \delta n_{\mathbf{p},\sigma}$ to give the change in the number of particles. However you will often see the calculations in text books done in terms of Eq. (12). At some stage when the sum over \mathbf{p} is done, it can be replaced by an angular average and an integration over $|\mathbf{p}|$ or over ε_p using the density of states $2N(0)$ (if the two spins are behaving in the same way)

$$\sum_{\mathbf{p},\sigma} \dots \rightarrow \frac{V}{(2\pi)^3} 2N(0) \int d\varepsilon_p \delta(\varepsilon_{\mathbf{p}} - \varepsilon_F) \int \frac{d\Omega}{4\pi} v_F \delta p_F(\hat{\mathbf{p}}) \dots \quad (13)$$

Specific Heat

Because of the quasiparticle interaction, the energy cost of adding an excitation at \mathbf{p} is no longer ε_p but is

$$\tilde{\varepsilon}_{\mathbf{p},\sigma} - \mu = \frac{\delta F}{\delta n_{\mathbf{p},\sigma}} = (\varepsilon_p - \mu) + \frac{1}{V} \sum_{\mathbf{p}',\sigma'} f(\mathbf{p}, \mathbf{p}'; \sigma, \sigma') \delta n_{\mathbf{p}',\sigma'}, \quad (14)$$

and depends on the other quasiparticles present. In general it may depend on \mathbf{p} and σ not only on $|p|$. Since the *counting* of states is exactly the same as in the noninteracting system, the entropy $S(\{\delta n_{\mathbf{p},\sigma}\})$ for a given quasiparticle distribution is the same as in the noninteracting system

$$S = -k_B \sum_{\mathbf{p},\sigma} [\delta n_{\mathbf{p},\sigma} \ln \delta n_{\mathbf{p},\sigma} + (1 - \delta n_{\mathbf{p},\sigma}) \ln(1 - \delta n_{\mathbf{p},\sigma})]. \quad (15)$$

This means that the equilibrium distribution (minimize $F - TS$ with respect to $\delta n_{\mathbf{p},\sigma}$) remains the Fermi distribution

$$\delta n_{\mathbf{p},\sigma}(T) = \frac{1}{e^{\beta(\tilde{\varepsilon}_{\mathbf{p},\sigma} - \mu)} + 1}, \quad (16)$$

with the difference that $\tilde{\varepsilon}_{\mathbf{p},\sigma}$ now depends self consistently on δn through Eq. (14).

Actually, in calculating the specific heat all these complications drop out. This is because summing over $|p|$, $\int dp p^2 \delta n_{\mathbf{p},\sigma}$ is $O(T^2)$. This means the interaction term in the energy is $O(T^4)$, whereas the single particle terms are $O(T^2)$. In addition $\mu = \varepsilon_F + O(T^2)$, and this change can again be neglected. The calculation proceeds just as for the noninteracting gas [see [Lecture 17](#) of [Ph127a](#), especially Eqs. (50-53)] but with the density of states Eq. (6), giving $C_V \propto T$, and

$$\frac{C_V}{C_{V\text{free}}} = \frac{m^*}{m}. \quad (17)$$

Compressibility

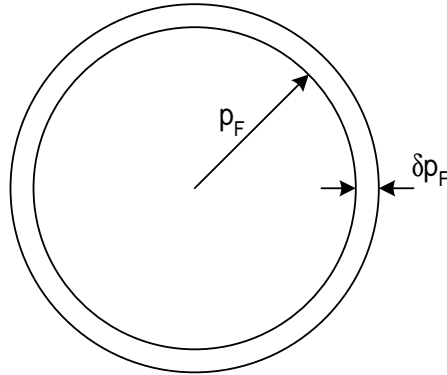


Figure 1: Construction to calculate the bulk modulus

The isothermal bulk modulus (the inverse of the compressibility), is

$$K_T = n \left(\frac{\partial P}{\partial n} \right)_{T,V}. \quad (18)$$

with n the density (number of particles per volume). Using the Gibbs-Duhem expression

$$d\mu = n^{-1}dP - sdT \quad (19)$$

this can be written

$$K_T = n^2 \left(\frac{\partial \mu}{\partial n} \right)_{T,V}. \quad (20)$$

The speed of sound s is given by

$$s^2 = \frac{K_T}{nm} = \frac{n}{m} \left(\frac{\partial \mu}{\partial n} \right)_{T,V}. \quad (21)$$

To calculate $\delta\mu/\delta n$ we start from a reference ground state, and put $\delta n_{\mathbf{p},\sigma} = 1$ in a shell at p_F of width δp_F so that the change in the total number of particles is $\sum \delta n_{\mathbf{p}} = V\delta n$ (see Fig. 1). The extra energy of adding a quasiparticle at $p_F + \delta p_F$ is then $\delta\mu$ which from Eq. (14) is

$$\delta\mu = v_F \delta p_F + \frac{1}{V} \sum_{\substack{\mathbf{p},\sigma \\ \text{shaded}}} f_{\mathbf{p},\mathbf{p}'}^{(s)} (\delta n_{\mathbf{p}',\uparrow} + \delta n_{\mathbf{p}',\downarrow}) \quad (22)$$

Obviously the spin antisymmetric part of $f_{\mathbf{p},\mathbf{p}'}$ does not contribute. Similarly only the $l = 0$ component contributes, and so

$$\delta\mu = v_F \delta p_F + \frac{F_0^{(s)}}{2N(0)} \left[\frac{1}{V} \sum_{\substack{\mathbf{p},\sigma \\ \text{shaded}}} (\delta n_{\mathbf{p},\uparrow} + \delta n_{\mathbf{p},\downarrow}) \right]. \quad (23)$$

The term in [] is just δn , which is also $2N(0)v_F\delta p_F$, so

$$\delta\mu = \frac{\delta n}{2N(0)} (1 + F_0^{(s)}). \quad (24)$$

Thus

$$\frac{\partial \mu}{\partial n} = \left(\frac{\partial \mu}{\partial n} \right)_{\text{free}} \frac{1 + F_0^{(s)}}{m^*/m} \quad (25)$$

with $(\partial\mu/\partial n)_{\text{free}} = \pi^2 \hbar^3 / m p_F$ the value in the noninteracting gas of the same density. The compressibility and s^2 are therefore changed from the values of the noninteracting system by the factor $(1 + F_0^{(s)})/(m^*/m)$. The speed of sound is actually given by

$$s^2 = \frac{1}{3} \left(\frac{p_F}{m} \right)^2 \frac{1 + F_0^{(s)}}{m^*/m}. \quad (26)$$

Magnetic susceptibility

Similar arguments give the renormalization of the magnetic susceptibility. Now spins are flipped, so that to linear order in the magnetic field B we have $\sum_{\mathbf{p}} \delta n_{\mathbf{p},\uparrow} = -\sum_{\mathbf{p}} \delta n_{\mathbf{p},\downarrow}$. The Fermi sea of up spins with lower energy in the field increases at the expense of the down spins, so that the chemical potentials are equal. The final results for the susceptibility (you will drive this in the homework) is

$$\chi = \mu_B^2 \frac{2N(0)}{1 + F_0^{(a)}} = \chi_{\text{free}} \frac{m^*/m}{1 + F_0^{(a)}}. \quad (27)$$

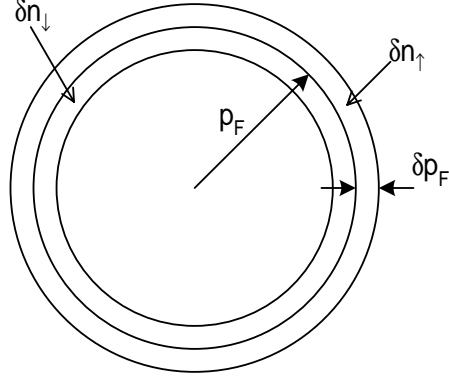


Figure 2: Construction to calculate the Fermi liquid corrections to the magnetic susceptibility.

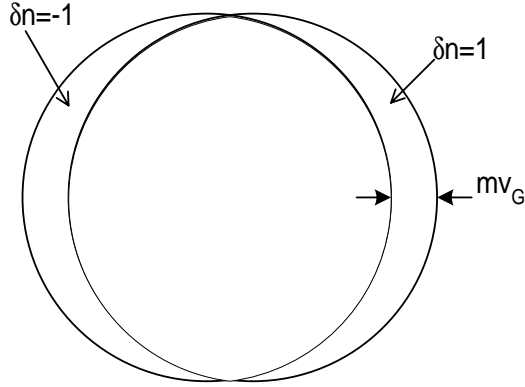


Figure 3: Change $\delta n_{\mathbf{p}}$ due to Galilean boost.

Effective Mass

For a translationally invariant system the requirement of Galilean invariance leads to the expression for the effective mass

$$\frac{m^*}{m} = 1 + \frac{1}{3} F_1^{(s)}. \quad (28)$$

Consider the change in energy of a quasiparticle at momentum \mathbf{p} due to a Galilean boost \mathbf{v}_G . We can evaluate this in two ways. The first way is a direct boost for an excitation of momentum \mathbf{p}

$$\delta E_{\mathbf{p}} = \mathbf{p} \cdot \mathbf{v}_G. \quad (29)$$

The second way is to construct the change in energy in the lab frame, from the change in the single particle and interaction terms coming from shifting all momentum by $m\mathbf{v}_G$

$$\delta E_{\mathbf{p}} = v_F \hat{\mathbf{p}} \cdot (m\mathbf{v}_G) + \frac{1}{V} \sum_{\mathbf{p}'} f_{\mathbf{p},\mathbf{p}'}^{(s)} (\delta n_{\mathbf{p}',\uparrow} + \delta n_{\mathbf{p}',\downarrow}) \quad (30)$$

where $\delta n_{\mathbf{p},\sigma}$ is the spin-symmetric change from the shift of the Fermi, see Fig. 3. Since $\delta n_{\mathbf{p},\sigma} \propto \cos \theta$ with θ the angle between $\hat{\mathbf{p}}$ and \mathbf{v}_G only the $l = 1$ component of $f_{\mathbf{p},\mathbf{p}'}^{(s)}$ will contribute, and we can make the

replacement

$$f_{\mathbf{p},\mathbf{p}'}^{(s)} \rightarrow \frac{F_1^{(s)}}{2N(0)} \hat{\mathbf{p}} \cdot \hat{\mathbf{p}}'. \quad (31)$$

Equating the two expressions for $\delta E_{\mathbf{p}}$ then gives

$$\mathbf{p} \cdot \mathbf{v}_G = v_F \hat{\mathbf{p}} \cdot (m\mathbf{v}_G) + \frac{F_1^{(s)}}{2N(0)} \hat{\mathbf{p}} \cdot \frac{1}{V} \sum_{\mathbf{p}'} \hat{\mathbf{p}}' (\delta n_{\mathbf{p}',\uparrow} + \delta n_{\mathbf{p}',\downarrow}) \quad (32)$$

We can calculate the sum

$$\frac{1}{V} \sum_{\mathbf{p}'} \hat{\mathbf{p}}' (\delta n_{\mathbf{p}',\uparrow} + \delta n_{\mathbf{p}',\downarrow}) = \langle \cos^2 \theta \rangle 2N(0) v_F m v_G \quad (33)$$

and $\langle \cos^2 \theta \rangle = 1/3$, so that dividing through by $p \simeq p_F$ and using $v_F = p_F/m^*$ gives

$$1 = \frac{m}{m^*} \left(1 + \frac{1}{3} F_1^{(s)} \right), \quad (34)$$

which is Eq. (28).

Parameters

The correction factors are often not small. For example for liquid He^3 the values are

	$\frac{m^*}{m}$	$F_0^{(s)}$	$F_0^{(a)}$	$F_1^{(s)}$	$\frac{s^2}{s_{\text{free}}^2}$	$\frac{\chi}{\chi_{\text{free}}}$
melting pressure	6.2	94.	-0.74	15.7	15.3	24.
zero pressure	3.0	10.1	-0.52	6.0	2.3	6.3

Particularly at the melting pressure, where the density is largest, some of the correction factors are very large, e.g. $F_0^{(s)}$ approaching 100. Nevertheless Fermi liquid theory is found to work very well: the qualitative behavior is the same as the noninteracting system ($C_V \propto T$, $\chi \rightarrow \text{const}$), with changed coefficients given by these parameters.

Further Reading

The Theory of Quantum Liquids, Vol. I by Nozieres and Pines is a readable reference on phenomenological Fermi liquid theory. *Statistical Mechanics, part 2* of the Landau and Lifshitz series is also good, but less readable. *Pathria* has a brief discussion in §10.8.

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