Chapter 6

Power Spectrum

The power spectrum answers the question "How much of the signal is at a frequency ω?". We have seen that periodic signals give peaks at a fundamental and its harmonics; quasiperiodic signals give peaks at linear combinations of two or more irrationally related frequencies (often giving the appearance of a main sequence and sidebands); and chaotic dynamics give broad band components to the spectrum. Indeed this later may be used as a criterion for identifying the dynamics as chaotic. Examples are shown in demonstration 1. These are all statements about the ideal power spectrum, if infinitely long sequences of continuous data are available to process. In practice there are always limitations of restricted data length and sampling frequency, and it is important to investigate how these limitations affect the appearance of the power spectrum.

6.1 Outline

If we did not have to worry about limitations in the data—i.e. we have a continuous time series $y(t)$ infinite in length—the power spectrum of the signal would be given simply by the Fourier transform: $P_{\text{ideal}}(\omega) \propto |\tilde{y}(\omega)|^2$ with

$$\tilde{y}(\omega) = \int_{-\infty}^{\infty} y(t)e^{-i\omega t}dt. \tag{6.1}$$

On the other hand we usually have a signal $y(t)$ measured over a finite interval $0 \le t \le T$ and with a finite sample rate so that we have N values of y is measured at intervals $t = integer \times \Delta$ (so that $T = N\Delta$). Then to estimate the power

spectrum of the signal we calculate the Fourier *series*

$$\tilde{y}_k = \sum_{j=0}^{N-1} y_j \exp\left(\frac{2\pi i j k}{N}\right) = \sum_0^{N-1} y(t_j) \exp\left(i\omega_k t_j\right) \tag{6.2}$$

where in the latter expression the discrete frequencies and times $\omega_k = 2\pi k/T$ and $t_j = j\Delta$ are introduced. (For a discrete time system of course the dynamics is given in terms of the index j.) For concreteness we take N to be even in the following. The power spectrum estimate is then (see ref. [1] for details)

$$P(\omega) \simeq \begin{cases} N^{-2} |\tilde{y}_0|^2 & \text{for} \quad \omega = 0 \\ N^{-2} \left(|\tilde{y}_k|^2 + |\tilde{y}_{N-k}|^2\right) & \text{for} \quad \omega = 2\pi k/T, \ k = 1, 2, \dots (\frac{N}{2} - 1) \\ N^{-2} \left|\tilde{y}_{N/2}\right|^2 & \text{for} \quad \omega = \pi N/T = \pi/\Delta \end{cases} \tag{6.3}$$

where we have used $|\tilde{y}_k| = |\tilde{y}_{-k}|$ for a real signal and $\tilde{y}_{-k} = \tilde{y}_{N-k}$ from (6.2). We will use P only for positive frequencies—the reason for the combination appearing for frequencies away from the two end points. The normalization is chosen so that the sum of $P(\omega)$ over the $\frac{N}{2} + 1$ frequencies ω_k is the mean square amplitude of the signal y_j. Other techniques for estimating the power spectrum also exist [1]. The question immediately arises of how the estimate $P(\omega)$ relates to the ideal spectrum $P_{\text{ideal}}(\omega)$.

To understand the full properties of this estimate of the spectrum it is useful to identify (6.2) as the Fourier *transform* of the modified time series $\bar{y}(t)$ defined by a rather complicated expression that separates out the different "imperfections" in the measurement:

$$\bar{y}(t) = [(y(t) \times H(t, T)) \otimes S(t, T)] \times S(t, \Delta) \quad . \tag{6.4}$$

Here $\otimes$ denotes a convolution, $H(t, \tau)$ is the top hat function, with value 1 for $0 < t < \tau$ and zero elsewhere, and $S(t, \tau)$ is the "periodic spike" function zero everywhere except at $t = n\tau$ with n any integer. Note that the convolution with $S(t, T)$ in (6.4) simply repeats the function $y(t) \times H(t, T)$ periodically with period T—which is what the Fourier series construction (6.2) effectively does—and the multiplication by $S(t, \Delta)$ is the sampling at the discrete intervals. The "ideal" power spectrum would be given simply by the Fourier transform $\tilde{y}(\omega)$ of $y(t)$.

Now we need three facts (see section 6.2 for further details):

(i) The Fourier transform of a product is the convolution of the Fourier transforms:

$$\text{if} \quad Y(t) = y_1(t) \times y_2(t) \quad \text{then} \quad \tilde{Y}(\omega) = \tilde{y}_1(\omega) \otimes \tilde{y}_2(\omega) \quad .$$

(ii) The Fourier transform of a convolution is the product of the Fourier transforms:

$$\text{if} \quad Y(t) = y_1(t) \otimes y_2(t) \quad \text{then} \quad \tilde{Y}(\omega) = \tilde{y}_1(\omega) \times \tilde{y}_2(\omega) \quad .$$

(iii) The Fourier transform of a periodic set of spikes with separation τ is also a periodic set of spikes and with separation in frequency $\frac{2\pi}{\tau}$; i.e. the Fourier transform of $S(t, \tau)$ with respect to t is just proportional to $S(\omega, \frac{2\pi}{\tau})$. This can be seen by noting the analogy to the diffraction pattern of a diffraction grating. Alternatively note that since $S(t, \tau)$ is periodic in t with period τ, its Fourier transform is nonzero only at the frequency $\frac{2\pi}{\tau}$ and its harmonics, and also since $S(t, \tau)$ is very non-sinusoidal (each peak has amplitude at all frequencies) we would expect a large amplitude in each of the harmonics.

So now we have

$$\bar{y}(\omega) \propto \left[\left(\tilde{y}(\omega) \otimes \tilde{H}(\omega, \frac{2\pi}{T}) \right) \times S(\omega, \frac{2\pi}{T}) \right] \otimes S(\omega, \frac{2\pi}{\Delta}) \qquad (6.5)$$

where $\left| \tilde{H}(\omega, \frac{2\pi}{T}) \right|$ is just the familiar sinch function $\left(\sin \frac{1}{2}\omega T \right) / \frac{1}{2}\omega$, which is maximum at $\omega = 0$, and has a width $\delta\omega \sim \frac{1}{T}$, but with oscillating tails falling off only as ω^{-1}. Figure 6.1 shows the effect of $\tilde{H}(\omega, \frac{2\pi}{T})$ on a pure frequency ω_s that is not an exact multiple of $\frac{2\pi}{T}$, so does not "fit" with the measurement period T. The line shows the functional form of the convolution in the [] in (6.5) for $\tilde{y}(\omega) = \delta(\omega - \omega_s)$ i.e. $\left| \tilde{H}(\omega - \omega_s) \right|$, and the points show the values at the discrete frequencies $n \times \frac{2\pi}{T}$. Note that the oscillations do not show up because the same scale $\frac{2\pi}{T}$ determines the oscillations and the discrete frequencies. (The values used in this plot are $\omega_s = .126\pi$, $T = 8$ so that $\frac{2\pi}{T} = \frac{\pi}{8}$, and $\Delta = 0.25$ so that the "Nyquist frequency" π/Δ is 4π.)

So we are told in (6.5) to "take the ideal Fourier transform $\tilde{y}(\omega)$, convolve (broaden) it with the resolution function $\tilde{H}$, sample the result at $\omega = integer \times \frac{2\pi}{T}$ and then superimpose the result repeatedly displaced by $\omega = integer \times \frac{2\pi}{\Delta}$". The last step means that we can restrict the range of ω to the Nyquist range $-\frac{\pi}{\Delta} < \omega \leq \frac{\pi}{\Delta}$ since other ranges are just duplications of this range, and that amplitudes
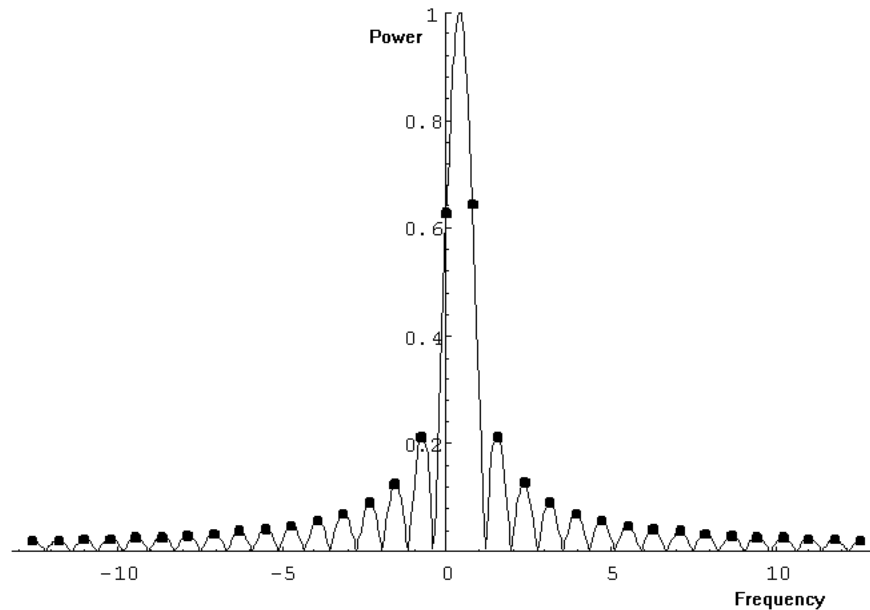
Figure 6.1: Power spectrum of $\exp(i\omega_s t)$ with frequency $\omega_s = 0.126\pi$ constructed from a time series of length $T = 8$ with steps $\Delta = 0.25$. (A complex signal is used for clarity. A real signal $\cos \omega_s t$ would be the superposition of this curve with the same curve centered around $-\omega_s$.)

in the ideal $\tilde{y}(\omega)$ outside this range will be folded back or "aliased" into the range (e.g. an amplitude at $\omega = \frac{3\pi}{2\Delta}$ will appear at $\omega = \frac{\pi}{2\Delta}$ due to the copy shifted to $\omega = -\frac{2\pi}{\Delta}$).

To get a better estimate of the spectrum it is customary to replace the discontinuous function H by a window function W, such as a tent, parabola or other shape (i.e. multiply the data by such a function). Since the high frequency tails of the Fourier transform are determined by the order of the derivative in which a discontinuity first appears—the power p of the ω^{-p} tail is the order of this derivative plus 1—this pushes the discontinuities to higher order derivatives and so makes the Fourier transform $\tilde{W}$ fall off more quickly with ω. Windowing is investigated in demonstration 2.

It is also important to make sure that the Nyquist frequency π/Δ is large enough (i.e. the sampling rate high enough), so that there is not significant power beyond this frequency, to minimize the effects of the aliasing. A final point is

that for nonperiodic signals the estimate $\tilde{y}_k$ is a very *noisy* estimate of the power spectrum i.e. different choices of which time interval T to measure will lead to a power spectrum rather different in the details. (You can see this from the figure: the largest value in the apparent power spectrum depends how close ω_s is to some integer multiple of $2\pi/T$). In fact the variance of $\tilde{y}_k$ is equal to the mean! To reduce the scatter take a number of power spectra (from different sets of measurements over intervals of length T) and *average* the *power* spectra.

The dependence of the appearance of the power spectrum on various parameters is illustrated in demonstration 3.

6.2 Details

This section is for reference only, and can be skipped on a first reading.

The convolution is defined as

$$y_1(t) \otimes y_2(t) = \int_{-\infty}^{\infty} y_1(t')y_2(t-t') \, dt' \quad , \tag{6.6}$$

and we define the Fourier transform as

$$
\begin{aligned}
\tilde{y}(\omega) &= \int_{-\infty}^{\infty} y(t)e^{-i\omega t} \, dt \\
y(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{y}(\omega)e^{i\omega t} \, d\omega
\end{aligned}
\quad . \tag{6.7}
$$

The "periodic spike" function $S(t, \tau)$ is defined formally as

$$S(t, \tau) = \sum_{n=-\infty}^{\infty} \delta(t - n\tau)$$

with δ the Dirac Delta function. The Fourier transform of $S(t, \tau)$ with respect to t is just $\frac{2\pi}{\tau} S(\omega, \frac{2\pi}{\tau})$ since we have

$$\tilde{S}(\omega) = \lim_{M \to \infty} \int_{-\infty}^{\infty} \sum_{n=-M}^{M} \delta(t - n\tau) \, e^{-i\omega t} \, dt = \lim_{M \to \infty} \sum_{n=-M}^{M} e^{-in\omega \tau} \quad . \tag{6.8}$$

giving

$$\tilde{S}(\omega) = \lim_{M \to \infty} \frac{\sin(M + \frac{1}{2})\omega\tau}{\sin\frac{1}{2}\omega\tau} \tag{6.9}$$

and then using the representation of a periodic sequence of delta functions

$$\lim_{M \to \infty} \frac{\sin(M + \frac{1}{2})x}{\sin \frac{1}{2}x} = 2\pi \sum_{n=-\infty}^{\infty} \delta(x - 2n\pi) \quad .$$

You can see this latter result by noting the value is very large, $2M + 1$, at $x = 2n\pi$ where the denominator goes to zero, falling to zero over the narrow distance π/M and the integral is

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(M + \frac{1}{2})x}{\sin \frac{1}{2}x} dx = 1 \quad .$$

(Plot the function for $M = 10$ if you do not believe this!) Also $\tilde{H}(\omega, \frac{2\pi}{T})$ is:

$$\tilde{H}(\omega, \frac{2\pi}{T}) = \int_{0}^{T} e^{-i\omega t} dt = e^{-i\omega T/2} \frac{\sin \frac{1}{2}\omega T}{\frac{1}{2}\omega} \quad .$$

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Bibliography

[1] *Numerical Recipes* by W.H. Press, B.P. Flannery, S.A. Teukolsky, and W.T. Vetterling, (Cambridge University Press, 2nd Edition) p. 542