Chapter 29

Quantum Chaos

What happens to a Hamiltonian system that for classical mechanics is chaotic when we include a nonzero $\hbar$? There is no problem in principle to answering this question: given a classical Hamiltonian, we can construct the quantum theory, for example the corresponding Schrodinger equation, and solve this (maybe numerically) in the standard ways. The question remains, however, how do the solutions to the quantum and classical problems compare, particularly the limit $\hbar \to 0$ where we would expect some "correspondence" between the results. Does the knowledge of the classical chaos help us understand the solutions to the quantum problem? Is there some remanence of "chaos" in the quantum solution?

The most obvious feature of the quantum problem is that a nonzero $\hbar$ leads to a finite splitting between the energy levels, so that the time dependence, given by a sum over the eigenstates $\sum_n A_n e^{-iE_n t/\hbar}$, becomes quasiperiodic rather than chaotic [1]. The energy splitting can be estimated from the volume of phase space available. For example for a particle of mass m in a two dimensional stadium of side L up to an energy $E \sim p^2/2m$ there are of order $mEL^2/\hbar^2$ eigenstates (the phase space volume is pL for each of two dimensions and the volume of four dimensional phase space per quantum state is $(2\pi\hbar)^2$, a statement of the uncertainty principle). Thus the average energy splitting is $\delta E \sim \hbar^2/mL^2$, and the frequency splitting $\delta E/\hbar$ goes to zero with $\hbar$. Thus quantum mechanics tends to quench the classical chaos, although the time scale for this to happen would be expected to diverge as $\hbar \to 0$.

There are interesting questions about the interface or crossover region, i.e. $\hbar$ "small" but nonzero, known as the semiclassical regime. Some of these are:

1. Bohr quantization, $E_n = (n+\phi)\hbar\omega$ with n any integer and ϕ some fractional

correction that depends on the system (e.g. $\phi = \frac{1}{2}$ for the harmonic oscillator), allows us to quantize *integrable* classical systems in the semiclassical limit, with ω the frequency of the periodic motion. Is it possible to quantize chaotic orbits in the semiclassical limit? Ideally we would like to be able to calculate energy levels and estimate wave functions, but a second best would be to learn less complete properties, e.g. statistics of energy spectra.

2. Does the difference between classically integrable and classically chaotic systems affect the quantum properties, and if so how?

3. What interesting effects arise as the $t \to \infty$ and $\hbar \to 0$ limits are taken?

We will address the first two issues for time independent Hamiltonians, and the second for the quantum version of the kicked rotor, where the Hamiltonian is periodic in time.

29.1 Energy Level Distribution

We are unlikely to be able to predict the exact energy levels of a classically chaotic system from knowledge of the classical motion—after all we can only understand the chaotic dynamics statistically. It makes sense therefore to ask statistical questions about the energy levels. Since the overall energy scale is set by details of the problem, the most basic question to ask is the statistics of the separations between adjacent energy levels on a scale small compared to the energy. Consider the distribution $P(s)$ where s is the separation of adjacent energy levels relative to the mean energy level spacing at that energy. A remarkably close agreement for this quantity is found between numerical or experimental measures of:

1. the energy levels of complicated quantum systems such as the energy levels of heavy nuclei in sectors of fixed quantum numbers such as angular momentum;

2. the eigenvalues of *random matrices* that could represent the Hamiltonian of time reversal symmetric physical systems (the Gaussian Orthogonal Ensemble, GOE);

3. the energy levels in the semiclassical limit of systems with chaotic classical orbits, such as the stadium and Sinai billiards (see chapter 28).

Furthermore the distribution $P(s)$ is well approximated by a functional form suggested by Wigner

$$P(s) = \frac{\pi}{2} s e^{-\pi s^2/4} \tag{29.1}$$

which is quite different from the assumption of a completely random distribution of the levels themselves, which would give a Poisson distribution $P(s) = e^{-s}$.

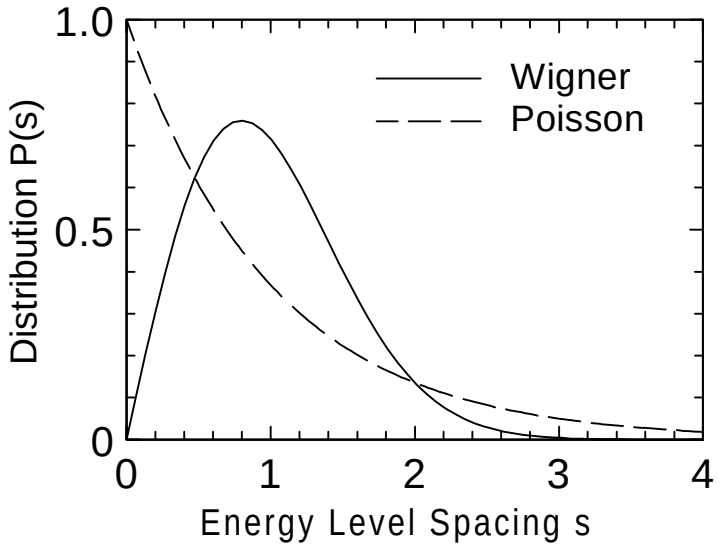

Figure 29.1: Wigner and Poisson distributions

The idea that the distribution of energy levels for complicated systems such as heavy nuclei might be given by random matrix theory goes back to Wigner in 1957. The appropriate random matrix should be Hermitian to correspond to a Hamiltonian, and then real for time reversal symmetry. The distribution of the Hamiltonians should be invariant under any orthogonal transformation (change of basis), hence the name GOE. The appropriate distribution turns out to be simply an independent Gaussian distribution for each element H_{jk} with $j \geq k$

$$p(\{H_{jk}\}) = C \exp[-A \sum_{jk} (H_{jk})^2] \tag{29.2}$$

with C set by the normalization. If time reversal symmetry is broken, for example by a magnetic field, the Hamiltonian is complex, the distribution should be invariant under unitary transformations (hence the name Gaussian Unitary Ensemble, GUE), and is given by taking independent Gaussian distributions for both real and imaginary parts of H_{jk} for $j \geq k$. There are exact expressions for $P(s)$ for $N \times N$ matrices for $N \to \infty$, but the "Wigner distribution" (29.1), which is the result for $N = 2$, is a good approximation to this. Note that there are no fit parameters, once the mean spacing is normalized out.

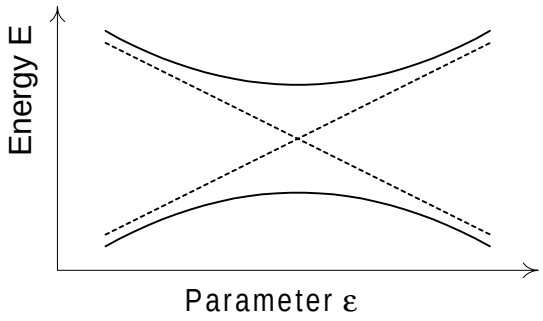

Figure 29.2: Energy level repulsion

The dramatic difference between the Poisson and Wigner distributions is the different dependences at small s: for the Wigner distribution $P(s) \to s$ for small s signifying a scarcity of levels spaced closely compared to the mean, whereas the Poisson distribution remains finite. This can be understood in terms of the well known phenomenon of level repulsion: two energy levels that appear to be going to cross as a parameter is varied appear to repel one another, so that no crossing occurs. This is easily motivated by the 2×2 case

$$H = \begin{bmatrix} \varepsilon & \delta \\ \delta & -\varepsilon \end{bmatrix} \tag{29.3}$$

where energy level crossing might be expected for $\varepsilon = 0$. In fact (see figure 29.2) the energy levels are $E = \pm\sqrt{\varepsilon^2 + \delta^2}$ so that for non-zero δ the levels stay apart by the distance δ, and for crossing *both* ε and δ have to be zero, with correspondingly reduced probability for random matrix elements. Of course level crossing is observed when the levels correspond to different symmetries, because

then the off-diagonal matrix elements are always zero (there is zero overlap integral of the wavefunctions with different symmetries). For the study of nuclei, the energy levels are first sorted by symmetry before the statistics are calculated.

The exact equivalence of the $P(s)$ for the random matrix problem and the classically chaotic system was conjectured by Bohigas, Giannoni and Schmit in 1984. The proof of this relationship would apparently have interesting ramifications in number theory, in particular the distribution of the zeroes of the Riemann zeta function. A proof has recently been claimed [3]. The distribution of separations for an *integrable* classical system on the other hand is given by the Poisson distribution.

29.2 The Quantum Rotor

The Hamiltonian for the unit mass undamped rotor with a periodic kick force $K\sin\theta$ with time period 1 (c.f. chapter 18, with a change of sign of K) is

$$H = \frac{p_\theta^2}{2} + K\cos\theta\sum_n \delta(t-n), \tag{29.4}$$

with p_θ the angular momentum $\dot{\theta}$. The quantum mechanical theory is given by the commutation rule $[\theta, p_\theta] = i\hbar$, equivalent to writing the momentum as $p_\theta = -i\hbar\partial/\partial\theta$. This gives the time dependent Schrodinger equation for the wave function $\psi(\theta, t)$

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\psi}{\partial\theta^2} + \left[K\cos\theta\sum_n \delta(t-n)\right]\psi. \tag{29.5}$$

Whereas the classical theory just depends on a single parameter, the quantum theory is defined by two parameters K and $\hbar$.

It is easy to construct the solutions by treating the kick and the time between the kicks separately. Integrating across the nth kick gives

$$\psi^+(\theta, n) = e^{-i(K/\hbar)\cos\theta}\psi^-(\theta, n). \tag{29.6}$$

where $\psi^+(\theta, n) = \lim_{\varepsilon\to 0}\psi(\theta, t = n+\varepsilon)$ etc. It is useful to introduce an angular momentum representation by Fourier transforming

$$\psi(\theta, t) = \frac{1}{\sqrt{2\pi}}\sum_m \tilde{\psi}_m(t)e^{im\theta}. \tag{29.7}$$

Clearly, between kicks each component evolves as

$$\tilde{\psi}_m(n+t) \sim e^{-i\hbar m^2 t/2}\tilde{\psi}_m^+(n), \quad t < 1 \tag{29.8}$$

These equations can then be combined to give the time evolution.

An interesting question to ask is what happens to the angular momentum as a function of time, i.e.

$$< p_\theta^2(t) > \hbar^2 \sum_m m^2 \left|\tilde{\psi}_m(t)\right|^2. \tag{29.9}$$

Classically, for large enough K this diffuses to large values

$$< p_\theta^2(t) > \sim Dt \tag{29.10}$$

with $D \approx \frac{1}{2}K^2$. Numerical evolution of the equations for the quantum rotor shows that for small time the behavior is the same as the classical prediction, but for $t > t^*$ (with t^* scaling as $(K/\hbar)^2$) the rate of increase flattens off and $< p_\theta^2 >$ saturates at a finite value. Thus the quantum effects "localize" the momentum.

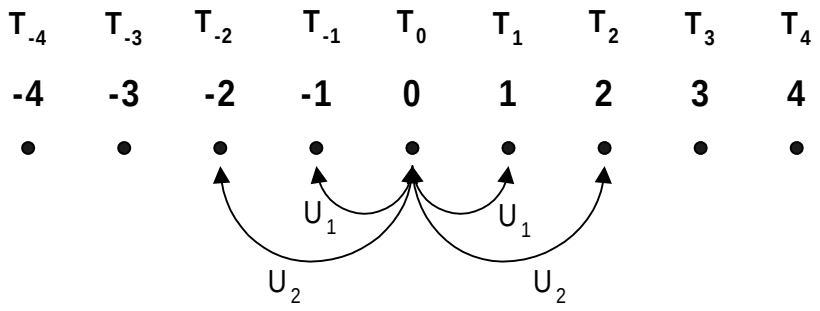

Figure 29.3: Hopping on the Electron Lattice

There is in fact an interesting analogy [2] to the localization of electrons in a random spatial potential, where again quantum effects cutoff the classical diffusive process (here in coordinate space rather than momentum space) giving an electrical insulator rather than a conductor. To make this connection note that since the Hamiltonian is periodic we can use Floquet theory to define a pseudo-energy $\hbar\omega$

$$\psi(\theta) = e^{-i\omega t}u(\theta, t) \tag{29.11}$$

with u satisfying the periodicity condition in time $u(\theta, t+1) = u(\theta, t)$. (If we turn off the kicking, ω reduces to the energy and the eigenstates are just the angular momentum states with energies $\hbar\omega_m = \hbar^2 m^2/2$.) Using the known behavior between the kicks and across each kick, and the periodic boundary conditions on u over the unit time interval, it is now possible to write the equation for the pseudo-eigenstates in the form

$$T_m \bar{u}_m + \sum_l U_l \bar{u}_{m+l} = (-U_0)\bar{u}_m \tag{29.12}$$

where $T_m = \tan\left(\omega - \hbar m^2/2\right)$ and $\bar{u}(\theta)$ (with Fourier transform $\bar{u}_m$) is defined in terms of the values $u^-(\theta)$ and $u^+(\theta)$ just before and after the kick

$$\bar{u} = \frac{1}{2}\left[u^+ + u^-\right] \tag{29.13}$$

$$= \frac{u^-}{1 - iU} = \frac{u^+}{1 + iU} \tag{29.14}$$

where we have used (29.6) and

$$U(\theta) = -\tan\left[(K/\hbar)\cos\theta\right] \tag{29.15}$$

and then $U_l = U_{-l}$ is the Fourier coefficient of $U(\theta)$. The analogy with the electron case (see Figure 29.3) is that $\bar{u}_n$ is the amplitude for the electron to be on the nth lattice site with site energy T_n, which for $\hbar/2\pi$ irrational is pseudo-random as n varies, U_l is the matrix element for hopping l steps to the right or left, and $-U_0$ is the eigenvalue (energy). For a random T_m it is known that each solution $\bar{u}_n$ is localized to some region of the lattice with exponential decay away from this region, rather than being extended, plane-wave like as in the nonrandom case.

These issues have application to experiments on the microwave ionization from highly excited hydrogen atoms. The high lying electronic states may be treated semiclassically, and the multiphoton ionization process can be analyzed in terms of the classical chaos analogous to the treatment of the quantum rotor.

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Bibliography

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