

Chapter 27

Hamiltonian Chaos: Introduction

The topic of Hamiltonian chaos is a whole course in itself, and we will only touch on a few highlights. For further information see Ott chapter 7, and review papers by Berry [1] and Helleman [2]. The ideal pendulum discussed in [chapter 2](#) is an example of a Hamiltonian system. In many ways this system is typical of the ones taught in elementary courses on mechanics: the orbits are periodic and can be constructed analytically or by perturbation theory. Other familiar examples are a collection of simple harmonic oscillators, or a planet revolving around the sun. All these systems are “integrable” and can be solved by various analytic means. However these examples taught are selected *because* they are solvable, rather than because they represent the typical behavior of Hamiltonian systems. Integrability is in fact the exception, and most Hamiltonian systems will show complex, chaotic dynamics.

A simple modification to the planetary problem illustrates the complexity of the motion in nonintegrable systems. Consider a “solar system” consisting of a binary pair of identical suns (coordinates \vec{R}_1, \vec{R}_2 , momenta \vec{P}_1, \vec{P}_2) of large mass M orbiting in the $x - y$ plane. Now add a planet of small mass m with an initial condition on the z axis above the centre of mass of the two suns and with a velocity in the z direction; by the symmetry of the system the small mass will remain on the z axis and its dynamics is defined by the coordinate z and z -momentum p_z . There are 6 independent dynamical variables: $X_1, Y_1, P_1, P_2, z, p_z$ (then $X_2 = -X_1$ etc. and the z coordinate and momenta of the heavy masses are fixed by the stationarity of the centre of mass). The energy and angular momentum are constants, leaving a 4-dimensional dynamical system. The dynamics of this system turns out to be quite complex. Consider the crossing times $\tau_1, \tau_2 \dots$ for which the “planet” crosses the plane of the orbit of the “suns”. It can be shown that for *any* chosen

sequence of numbers (e.g. random numbers) an initial condition of the 3 body system can be found such that the τ_i reproduce this sequence, with escape to infinity corresponding to the end of a finite sequence! Thus no matter how many τ_i are measured, no prediction can be made for the next τ in the dynamics.

The key difference between Hamiltonian and dissipative systems is that the dynamics of the former preserves volume elements in phase space, whereas for the latter volumes contract to zero. A consequence of the conservation of phase space volumes is that attractors do not exist in Hamiltonian systems, and we are left with the task of understanding the dynamics in the *whole* of phase space. An example is the 2d-circle map:

$$\begin{aligned} x_{n+1} &= x_n + \Omega + by_n - \frac{K}{2\pi} \sin 2\pi x_n \\ y_{n+1} &= by_n - \frac{K}{2\pi} \sin 2\pi x_n \end{aligned} \quad (27.1)$$

This map describes the dynamics of the periodically kicked rotor ([chapter 18](#)). For $\Omega = 0$ the equation also describes the motion of a ball bouncing on an oscillating surface. In both cases b measures the dissipation, and $b = 1$ for no dissipation. The determinant of the Jacobean of the map is

$$\begin{vmatrix} 1 + K \cos 2\pi x_n & b \\ K \cos 2\pi x_n & b \end{vmatrix} = b, \quad (27.2)$$

and the map is area preserving for $b = 1$ and dissipative for $b < 1$. The area preserving case with $\Omega = 0$ is called the standard map (and conventionally the sign of y is reversed)

$$\begin{aligned} x_{n+1} &= x_n + y_n + \frac{K}{2\pi} \sin 2\pi x_n \\ y_{n+1} &= y_n + \frac{K}{2\pi} \sin 2\pi x_n \end{aligned} \quad (27.3)$$

These ideas are illustrated in [demonstrations 1 and 2](#).

27.1 Formalism

27.1.1 Phase space

The phase space of a Hamiltonian system can be chosen as N “position” coordinates forming a vector \vec{q} and N conjugate “momentum” coordinates \vec{p} forming an $M = 2N$ dimensional phase space. The dynamics is determined by a Hamiltonian

$H(\vec{p}, \vec{q}, t) :$

$$\begin{aligned}\frac{d\vec{p}}{dt} &= -\frac{\partial H}{\partial \vec{q}}, \\ \frac{d\vec{q}}{dt} &= \frac{\partial H}{\partial \vec{p}}.\end{aligned}\tag{27.4}$$

If H has no explicit time dependence i.e. $H = H(\vec{p}(t), \vec{q}(t))$ then H is a constant of the motion $dH/dt = 0$ that we identify with the energy.

Hamiltonian dynamics is volume preserving

$$\vec{\nabla} \cdot \vec{V}_{ph} = \frac{\partial}{\partial \vec{p}} \frac{d\vec{p}}{dt} + \frac{\partial}{\partial \vec{q}} \frac{d\vec{q}}{dt} = 0.\tag{27.5}$$

A stricter property (from which volume preserving follows) is the symplectic property: if we have 3 orbits infinitesimally separated by $(\delta\vec{p}, \delta\vec{q})$ and $(\delta\vec{p}', \delta\vec{q}')$ then

$$\frac{d}{dt} (\delta\vec{p}\delta\vec{q}' - \delta\vec{q}\delta\vec{p}') = 0,\tag{27.6}$$

or in integral form

$$\frac{d}{dt} \oint_{\gamma} \vec{p} \cdot d\vec{q} = 0\tag{27.7}$$

where γ is a closed contour that evolves with the flow. (For a two dimensional phase space the symplectic property is equivalent to the property of area preserving.)

27.1.2 Canonical transformations

A transformation to a new set of coordinates (\vec{p}', \vec{q}') that leads to equations of motion in the same form as (27.4) is said to be canonical. A canonical transformation can be formed from a generating function $S(\vec{p}', \vec{q}, t)$ that is a function of the *old* position and *new* momentum coordinates. The change of variables is then given by the implicit equations

$$\vec{q}' = \frac{\partial S(\vec{p}', \vec{q}, t)}{\partial \vec{p}'}, \quad \vec{p} = \frac{\partial S(\vec{p}', \vec{q}, t)}{\partial \vec{q}}.\tag{27.8}$$

It can be checked that the new variables satisfy the symplectic condition. The new Hamiltonian is given by

$$H'(\vec{p}', \vec{q}', t) = H(\vec{p}, \vec{q}, t) + \partial S / \partial t.\tag{27.9}$$

27.1.3 Reduction to maps

For a time independent N -dimensional Hamiltonian system the dynamics is confined to a $2N - 1$ dimensional constant energy surface. The intersection with a constant q_0 plane gives a $2N - 2$ dimensional surface and the dynamics is given by a map on this surface. For a time periodic N -dimensional Hamiltonian, period τ , the phase space coordinates (\vec{p}, \vec{q}) at times $n\tau$ are related by a time independent, $2N$ -dimensional map (c.f. the periodically kicked pendulum). In either case it may be shown that the map is symplectic.

The symplectic property shows that the eigenvalues giving the behavior of a small displacement from a fixed point of the map must come in $(\lambda, 1/\lambda)$ pairs (the Lyapunov exponents come in \pm pairs). This can be shown by choosing the 3 trajectories in the definition of the symplectic property to be the fixed point and trajectories along the eigenvectors.

For a two dimensional map there are three possible types of behavior typical near a fixed point:

1. Hyperbolic fixed point: $\lambda > 1, 1/\lambda < 1$ with λ real and positive. Trajectories near the fixed point are hyperbolic, with growth in one direction i.e. the fixed point is “linearly unstable”
2. Elliptic fixed point: $\lambda, 1/\lambda = e^{\pm i\phi}$. The trajectories near the fixed point are elliptic and points remain in the vicinity of the fixed point, and the fixed point might be called “linearly stable”.
3. Hyperbolic with reflection: $|\lambda| > 1, 1/|\lambda| < 1$ with λ real and negative. As type 1, but with an alternation of the sign of the displacement from the fixed point at each iteration.

27.2 Constants of the motion and integrability

The Poisson bracket $\{ \}$ of two quantities (c.f. commutator in quantum mechanics) is defined as

$$\{A, B\} = \frac{\partial A}{\partial \vec{q}} \cdot \frac{\partial B}{\partial \vec{p}} - \frac{\partial A}{\partial \vec{p}} \frac{\partial B}{\partial \vec{q}}. \quad (27.10)$$

A quantity $f(\vec{p}, \vec{q})$ that has zero Poisson bracket with the Hamiltonian is a constant of the motion:

$$\frac{df}{dt} = \frac{\partial f}{\partial \vec{q}} \frac{d\vec{q}}{dt} + \frac{\partial f}{\partial \vec{p}} \frac{d\vec{p}}{dt} = \{f, H\} = 0. \quad (27.11)$$

A time independent Hamiltonian for a $2N$ dimensional phase space is said to be integrable if there are N independent constants of the motion f_i that are in “involution” i.e. $\{f_i, f_j\} = 0$. One of the constants is the energy. The motion lies on the surface of an N -dimensional torus. A canonical transformation of variables can be made to “action-angle” variables

$$(\vec{p}, \vec{q}) \Rightarrow (\vec{I}, \vec{\theta}) \quad (27.12)$$

with $0 \leq \theta < 2\pi$ such that the Hamiltonian is only a function of the \vec{I} , i.e. $H = H(\vec{I})$. The \vec{I} are then constants of the motion, and the $\vec{\theta}$, corresponding to the angles around the torus, advance at a constant rate:

$$\frac{d\vec{I}}{dt} = 0; \quad \frac{d\vec{\theta}}{dt} = \vec{\omega}(\vec{I}) = \frac{\partial H}{\partial \vec{I}}. \quad (27.13)$$

The action variables can be related to the original (\vec{p}, \vec{q}) through

$$I_i = \frac{1}{2\pi} \oint_{C_i} \vec{p} \cdot d\vec{q} \quad (27.14)$$

with the integration around the i th axis of the torus.

A simple example of an integrable system is a set of N harmonic oscillators (of unit mass)

$$H = \sum_j \frac{1}{2} (p_j^2 + \omega_j^2 q_j^2). \quad (27.15)$$

The transformation to action-angle variables is

$$p_j = (2I_j \omega_j)^{1/2} \cos \theta_j, \quad q_j = (2I_j / \omega_j)^{1/2} \sin \theta_j, \quad (27.16)$$

and then the Hamiltonian is

$$H(\vec{I}, \vec{\theta}) = \sum_j \omega_j I_j. \quad (27.17)$$

The I_j are constants of the motion and the θ_j increase at a constant rate

$$\theta_j(t) = \omega_j t + \theta_j^{(0)} \quad (27.18)$$

and in this special case the frequencies are independent of \vec{I} . Substituting into (27.16) gives the usual limit cycle dynamics.

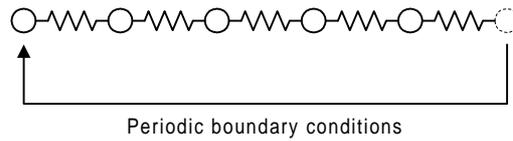


Figure 27.1: An Integrable System

A physical situation leading to (27.15) is a set of N equal masses connected by harmonic springs (force F proportional to extension x) and with periodic boundary conditions (the N th mass connected back to the first one). The “normal modes” j are then just Fourier modes. This is a simple model of a lattice of atoms. Since the system is integrable, the dynamics is non-ergodic, and there is no equilibration—energy put into one mode will stay there and will not be distributed according to equipartition. In 1955 Fermi, Ulam and Pasta investigated numerically the question of equilibration with anharmonic springs e.g. $F(x) = ax + bx^p$ with $p = 2$ or 3 , expecting equilibration on a short time scale determined by b . The system is now nonintegrable, but they found that energy injected in one linear mode, although apparently becoming distributed over many modes, actually recoheres back into the original mode to a surprising degree, so that the equilibration process is much slower than expected. It is interesting to note that a *particular* form of nonlinear force law $F(x) = a(1 - \exp(-bx))$ yields an integrable system—known as the Toda lattice—where the N constants of the motion are complicated functions of the position and momentum coordinates: not all integrable systems are trivial!

The question of integrability in continuous systems represented by partial differential equations is particularly interesting: for equations in one space variable

numerous integrable systems are known leading to the fascinating and important phenomenon of solitons, and the highly non-trivial mathematics of the “inverse scattering” formalism.

27.3 Nonintegrable systems

What happens if we perturb an integrable Hamiltonian H_0

$$H = H_0 + \varepsilon H_1? \quad (27.19)$$

In general H will not be integrable for any nonzero ε . The dynamics of H_0 consists of motion on N -tori determined by the values of the N constants of the motion. What is the fate of these tori for the non-integrable Hamiltonian H as ε is increased? This leads to a fascinatingly rich set of phenomena and mathematics, which will be illustrated numerically here, and discussed more generally, albeit qualitatively, in the [next chapter](#).

The fate of the tori depends on the frequency vector $\vec{\omega}$, and in particular on the ratios of the values of the components. For an $N = 2$ time independent Hamiltonian (a 4 dimensional phase space leading to a 2-dimensional map) this is given by a single winding number W , the ratio of the frequencies about the two axes. For a periodically driven $N = 1$ Hamiltonian, again leading to a two dimensional map, the single winding number is the frequency of the map on the Poincaré section.

The standard map is integrable for $K = 0$ with y a constant of the motion and the “tori” (here a limit cycle) given by

$$x_n = x_{n+1} + y. \quad (27.20)$$

For $K \neq 0$ the map is nonintegrable, and the map can be used to illustrate the following behavior:

1. The tori corresponding to rational winding numbers break down leading to frequency locking and chaos.
2. A mathematical theory known as KAM (Kolmogorov-Arnold-Mosur) theory shows that most (i.e. the complement of a set of measure zero in the winding number) irrational tori survive for small perturbations $\varepsilon \rightarrow 0$.

3. The “very irrational” tori survive longest, and the break down of the last irrational torus (given by a winding number equal to the Golden Mean $W = \frac{1}{2}(\sqrt{5} - 1)$) has a scaling structure.
4. For two dimensional maps the tori provide barriers against the dynamics exploring the whole phase space. The break down of the last of the original tori provides an avenue for this exploration to occur, which is then described by “Arnold diffusion”. (In the case of a two dimensional map describing a periodically driven $N = 1$ Hamiltonian system this may correspond to the diffusion of the system to large energies.)
5. The elliptic fixed points of the map may undergo an infinite sequence of period doubling bifurcations to chaos with a new set of universal constants δ and α .

(see [demonstrations 3-7](#)).

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Bibliography

- [1] M.V. Berry, “*Regular and Irregular Motion*”, AIP Conference Proceedings **46**, 16 (1978)
- [2] R.H.G. Helleman, “*Self generated chaotic behavior in nonlinear mechanics*”, in *Fundamental Problems in Statistical Mechanics* (North Holland)