Chapter 26

Shadowing

As we saw at the beginning of this course, the exponential growth of errors iterating a chaotic dynamical system implies that a computer generated trajectory from some initial condition will rapidly diverge from the true orbit due to truncation errors or approximations in the numerical integration scheme, so that after a relatively short time the computer generated orbit will have no correlation with the true orbit. This means, for example, that the relationship of numerical modeling of a physical system to the actual behavior is unclear. Our confidence in the modeling is partially rescued, at least in chaotic systems of moderate dimension, by the phenomenon of shadowing Shadowing is the existence of a *true* trajectory that remains close to the numerically produced trajectory (called the *pseudo-trajectory*) for very long times, This true trajectory will not in general be the one with the same initial condition as the numerical trajectory.

The phenomenon of shadowing has been known for a long time for hyperbolic systems [1]. Recent work has addressed the question for nonhyperbolic physical attractors.

26.1 Shadowing in Hyperbolic Systems

The idea of shadowing is surprisingly simple. Suppose we have a one dimensional chaotic system, e.g. a map of the unit interval in x. Errors will grow under forward iteration, so that a numerical trajectory will diverge from the true trajectory. However given the position p_n of the pseudo-trajectory at the nth step, we can imagine iterating backwards to find the preimages of this point. Since the map is contracting under inverse iterations, the error decays for backwards trajectories, and the trajectory remains close to the backwards iteration of the *true* trajectory starting at p_n. This procedure applies to a hyperbolic system in higher dimensions, since the expanding and contracting directions are consistent and separate. The shadowing trajectory is found by integrating forward from the initial point the contracting directions, and integrating backwards from the final point of the pseudo-trajectory the expanding directions.

The existence of the shadowing trajectory near the pseudo-trajectory is shown by the following argument [2] (we take the case of a two dimensional map F for simplicity).

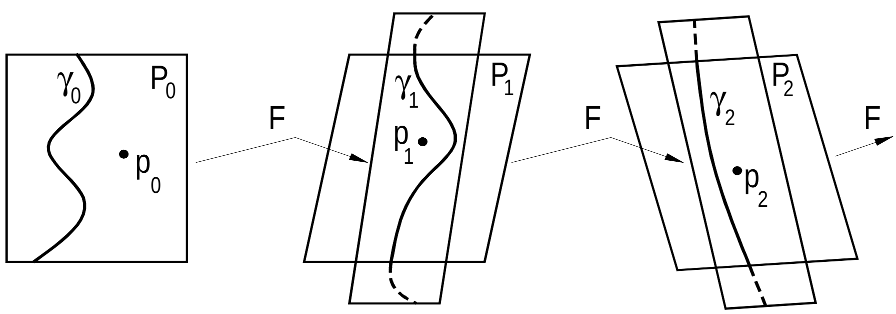

Figure 26.1: Construction of the first few parallelograms P_i.

Surround the points on the pseudo-trajectory p_j and p_{j+1} by parallelograms P_j and P_{j+1}, with sides given by pieces of the stable and unstable manifolds, chosen so that $F(P_j)$ straddles P_{j+1} as shown in figure 26.1. If the one step error is less than δ then the sides of the parallelograms can be restricted to a few times δ. Now consider a closed curve γ_0 in P_0 running from one of the contracting sides to the other. Then $F(\gamma_o)$ must contain a closed curve γ_1 that lies wholly within P_1 and runs from one contracting side to the other. Continue this to generate the sequence of curves γ_j completely in P_j. Now choose a point x_n on γ_n. Then this point is close to p_n (within a few δ). Also, each preimage $F^{j-n}(x_n)$ lies on γ_j and is therefore close to within a few δ of p_j. This shows the existence of the true trajectory within the confining parallelograms.

An approximation to the true trajectory can be constructed by the "refinement" technique [2]. Suppose that π_{n+1} is the one step error due to truncation or noise

$$\pi_{n+1} = p_{n+1} - f(p_n) \tag{26.1}$$

where it is assumed that $|\pi_{n+1}| < \delta$. The refined orbit is $\{\tilde{p}_n\}$ with $\tilde{p}_{n+1} = F(\tilde{p}_n)$ where

$$\tilde{p}_n = p_n + \Phi_n \tag{26.2}$$

defines the correction to be found. The equation satisfied by Φ is

$$\begin{aligned}\Phi_{n+1} &= \tilde{p}_{n+1} - p_{n+1} \\ &= F(\tilde{p}_n) - F(p_n) - \pi_{n+1}.\end{aligned} \tag{26.3}$$

Assuming Φ_n is small, the difference is given by linearizing, so that

$$\Phi_{n+1} = J_n \Phi_n - \pi_{n+1} \tag{26.4}$$

where J_n is the linearized map DF at p_n. Now write Φ_n and π_n in terms of components along the stable and unstable directions s_n and u_n at p_n, i.e. $\Phi_n = \alpha_n u_n + \beta_n s_n$ and $\pi_n = \eta_n u_n + \zeta_n s_n$. Substitute into (26.4) using the result that $J_n u_n$ is along u_{n+1} etc., and equate components along u_{n+1} and s_{n+1}:

$$\begin{aligned}\alpha_{n+1} &= |J_n u_n| \alpha_n - \eta_{n+1} \\ \beta_{n+1} &= |J_n s_n| \beta_n - \zeta_{n+1}\end{aligned} \tag{26.5}$$

These equations are solved recursively, calculating α_n (the coefficients in the unstable direction) backwards from the last point $n = N$

$$\alpha_n = (\alpha_{n+1} + \eta_{n+1}) / |J_n u_n|, \quad \alpha_N = 0 \tag{26.6}$$

and β_n forwards from $n = 0$

$$\beta_{n+1} = \beta_n |J_n s_n| - \zeta_{n+1}, \quad \beta_0 = 0. \tag{26.7}$$

Iterating the refinement step gives better and better approximations to the true trajectory. Note that $|J_n u_n|^{-1}$ and $|J_n s_n|$ are less than some ρ with $0 < \rho < 1$ for a hyperbolic attractor, so that these are convergent iterations.

26.2 Shadowing in Nonhyperbolic Systems

The difficulty arises in nonhyperbolic orbits because the growth rates are not bounded away from unity, and the angle between the stable and unstable subspaces is not bounded away from zero. How these quantities affect the construction of a shadowing orbit is shown by the "shadowing theorem" of Sauer and Yorke [3], which is roughly the following. Suppose $\{p_n\}$ is a pseudo-orbit with step error δ. Also r_n is an upper bound for the expansion rate of the linear map DF in the stable subspace S_n and t_n is an upper bound for the expansion of DF^{-1} in the unstable subspace U_n (actually S_n and U_n are defined as approximate stable and unstable subspaces at the point p_n on the pseudo-trajectory). In addition θ_n is the angle between stable and unstable subspaces. Define iteratively

$$\begin{aligned} D_n &= \csc\theta_n + t_n D_{n+1}, \quad D_N = 0 \\ C_n &= \csc\theta_n + r_{n-1} C_{n-1}, \quad C_0 = 0. \end{aligned} \tag{26.8}$$

Then if $\max\{C_j, D_j\} < A/\sqrt{\delta}$ (where A is a number that depends on the dimension of the map and the maximum size of the first and second derivatives of F and F^{-1}) there exists a true orbit $\{x_n\}$ of F with $|x_n - p_n| < \sqrt{\delta}$.

This result tells us, for a particular orbit of length N that comes dangerously close to a nonhyperbolic point so that r_n, t_n, and $\csc\theta_n$ may become large, how bad the shadowing gets. Eventually, for large enough N the orbit will become so close to a nonhyperbolic point that the error becomes comparable to the size of the attractor and shadowing fails completely. This is called a *glitch*.

An estimate for the shadowing time is given by parametrizing the nonhyperbolic behavior in terms of the fluctuations of the *finite time* Lyapunov exponents, given by the usual definition but averaging over a large but finite number of iterations (e.g. 100). A finite time Lyapunov exponent fluctuating about zero is a signature of nonhyperbolic behavior. The mean m of the exponent acts to exponentially quench errors (iterate backwards or forwards as appropriate), but fluctuations about zero due to the variance σ^2 will lead to the growth of errors. Sauer et al. [4] suggest that $y = \log d$, with d the distance between the pseudo-trajectory and the shadowing trajectory, will evolve as a biased random walk with a probability distribution $P(y)$ described by a diffusion equation with drift

$$\frac{\partial P}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial y^2} + |m|\,\frac{\partial P}{\partial y} \tag{26.9}$$

together with a reflecting barrier at $y = \log\delta$ since there is an error of this size at each step. Here σ is the single step variance of the Lyapunov exponent near zero, calculated as $\sqrt{T}$ times the variance of the time-T Lyapunov exponent. Note that the mean m leads to a drift of $\log d$ to smaller (more negative) values, whereas the diffusion term depending on σ^2 will lead to spreading so that the tail will eventually extend to large values. The validity of the diffusion description is confirmed numerically by an observed stationary probability distribution that is consistent with the exponential distribution $P(y) \sim e^{-2|m|y/\sigma^2}$ (with a cutoff at $y = \log \delta$). This leads to a power law distribution of d i.e. $P(d) \sim d^{-2|m|/\sigma^2}$. The average shadowing time is given by the average time for y to diffuse to values of order unity (the size of the attractor). From the diffusion equation this is of order[1]

$$<\tau> \sim \delta^{-2|m|/\sigma^2}. \tag{26.10}$$

Notice when $|m|/\sigma^2$ is close to zero the shadowing time depends very weakly on the step error δ, so that increasing the accuracy of the simulation does little to improve the time for which the pseudo-orbit approximates the true orbit.

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[1] I have only found a rather roundabout argument to prove this result. Equation (26.9) describes the diffusive motion of a "particle" of mobility μ in a potential $|m|y/\mu$ (with a hard wall at $y = \log \delta$) and with diffusion constant $D = \frac{1}{2}\sigma^2$. The probability will be concentrated at small y near $\log \delta$. We can estimate the time τ for a particle to reach y_0 of order unity through a Boltzmann expression (the exponential will dominate any prefactors, as in calculating thermally activated escape rates)

$$\tau^{-1} \sim \exp(-\frac{|m|y_0}{\mu k_B T}) / \int_{\log \delta}^{\infty} dy \exp(-\frac{|m|y_0}{\mu k_B T}),$$

which gives (again ignoring many prefactors)

$$\tau \sim \exp\left(\frac{-m \log \delta}{\mu k_B T}\right).$$

But the Einstein relation gives

$$\mu k_B T = D = \frac{1}{2}\sigma^2$$

and so the result.

Bibliography

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