Chapter 16

Renormalization Group Theory

In the previous chapter a procedure was developed where higher order 2^n cycles were related to lower order cycles through a "functional composition and rescaling" procedure. Based on this pictorial approach we are led to a formal understanding of the period doubling route to chaos. This renormalization approach was conjectured by Feigenbaum [1], based on a similar qualitative analysis, and was later proved rigorously by Collet and Eckman [2] for a power law maximum $\left|x-\frac{1}{2}\right|^{1+\varepsilon}$ with small ε, and using a rigorous numerical proof by Lanford [3] for the quadratic maximum.

It is convenient to shift coordinates $x \to x-\frac{1}{2}$ so that the maximum is at $x=0$. The map is now of the interval $-\frac{1}{2} \leq x \leq \frac{1}{2}$ and is given by

$$x_{n+1} = -\frac{1}{2} + a\left(\frac{1}{4} - x_n^2\right), \tag{16.1}$$

(see figure 16.1).

16.1 The fixed point

Define an operator T that preforms the functional composition ("iteration") and rescaling:

$$T[f](x) = -\alpha f\left(f\left(-\frac{x}{\alpha}\right)\right). \tag{16.2}$$

Note that T operates on the function f.

The universal behavior of the sequence of subharmonic bifurcations is understood from the following two statements:

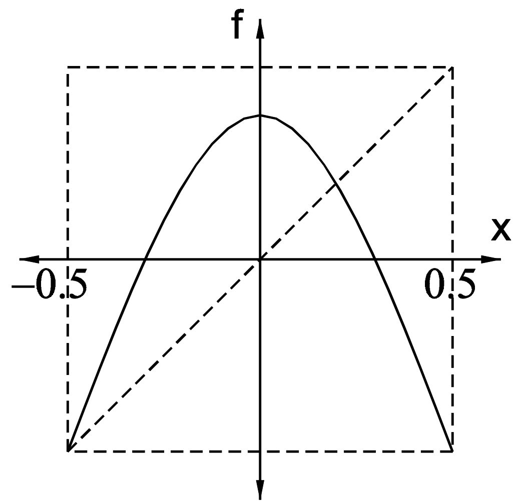

Figure 16.1: Map of the unit interval shifted to have maximum at $x = 0$.

1. The operation T has a fixed point solution $g(x)$ *for a particular value of* α, i.e. $T[g] = g$ or

$$g(x) = -\alpha g\left(g\left(-\frac{x}{\alpha}\right)\right). \tag{16.3}$$

   To completely define the solution g we must fix the nature of the maximum to be quadratic at $x = 0$, and to set the overall scale (since if $g(x)$ is a solution $\mu g(x/\mu)$ is also easily seen to be a solution) we choose $g(0) = 1$.

2. Linearizing about the fixed point of T yields a *single* unstable direction (eigenvector) with eigenvalue δ, i.e. writing

$$f(x) = g(x) + \phi(x) \tag{16.4}$$

   with ϕ small, and defining the linearized operator

$$L[\phi] = T[g + \phi] - T[g] \tag{16.5}$$

   we must have

$$L\left[\phi^{(i)}\right](x) = \lambda^{(i)}\phi^{(i)}(x) \tag{16.6}$$

defining the eigenvectors $\phi^{(i)}$ and the eigenvalues $\lambda^{(i)}$. Then ordering the eigenvalues in decreasing sequence we must have $\lambda^{(1)} = \delta > 0$ (and we then write $\phi^{(0)}$ as $h(x)$), and $\lambda^{(i)} < 0$ for $i > 1$. Explicitly, we have for the linearization

$$T\left[g+\phi\right] = -\alpha\left(g+\phi\right)\left(g\left(-\frac{x}{\alpha}\right)+\phi\left(-\frac{x}{\alpha}\right)\right). \tag{16.7}$$

Then

$$L\left[\phi\right](x) = -\alpha\left\{\phi\left(g\left(-\frac{x}{\alpha}\right)\right)+g'\left(g\left(-\frac{x}{\alpha}\right)\right)\phi\left(-\frac{x}{\alpha}\right)\right\}.$$

It may be worth emphasizing that we are looking at the fixed point and linearization of the *operation* T which acts on the space of *functions*, and *not* the fixed point and linearization of the *map* which acts on a point in the unit interval.

16.2 Evaluation of α and δ

The numbers α and δ are defined by this abstract procedure, without any reference to a dynamical system. This can be illustrated with a very crude approximation.

First for the fixed point equation we approximate $g(x)$ as

$$g(x) = 1 + bx^2 + \cdots \tag{16.8}$$

and ignore all the higher order terms represented by the $\cdots$. Then

$$g\left(g\left(-\frac{x}{\alpha}\right)\right) = 1 + b\left(1 + bx^2/\alpha^2\right)^2. \tag{16.9}$$

This is expanded up to $O(x^2)$ so that the fixed point equation becomes

$$1 + bx^2 = -\alpha\left[1 + b + (2b^2/\alpha^2)x^2\right]. \tag{16.10}$$

Equating coefficients then gives $b = -\alpha/2$ and

$$\alpha = 1 + \sqrt{3} \simeq 2.73 \tag{16.11}$$

(choosing the positive root for α since b must be negative).

The value of δ is evaluated by an even cruder approximation for the eigenfunction $h(x)$: we simply take the first term in a Taylor expansion, i.e. $h(x) \simeq 1$, and demand that the linearization equation

$$-\alpha\left\{h\left(g\left(-\frac{x}{\alpha}\right)\right)+g'\left(g\left(-\frac{x}{\alpha}\right)\right)h\left(-\frac{x}{\alpha}\right)\right\}=\delta h(x) \qquad (16.12)$$

be satisfied at $x=0$ (where this approximation is best). This gives

$$-\alpha\left[g'(g(0))+1\right]=\delta. \qquad (16.13)$$

But $g(0)=1$ and $g'(1)=2b=-\alpha$ using (16.8) for g. So

$$\delta \simeq \alpha^2-\alpha \simeq 4.73. \qquad (16.14)$$

These are very crude estimates just to show that the numbers are defined by statements 1 and 2. Better estimates (e.g. keep more terms in the power series expansions) give

$$\alpha=-2.502807876\ldots, \quad \delta=4.6692016\ldots. \qquad (16.15)$$

16.3 Universal map functions

As all other "directions" around the fixed point contract, we can move out from the fixed point along the unstable "direction" defined by the eigenvector h. Define

$$g_r(x)=g(x)+\delta^{-r}h(x) \qquad (16.16)$$

for large r. Then

$$Tg_r=T\left[g+\delta^{-r}h\right]=g+\delta^{-(r-1)}h=g_{r-1}, \qquad (16.17)$$

where the second equality comes from the linearization near the fixed point and the definition of h as the eigenvector with eigenvalue δ. Eventually as we repeat this operation the function g_r for decreasing r moves away from the vicinity of the fixed point and the linearization procedure fails. However we can continue to define g_r at smaller r through the operation of T:

$$g_{r-1}=Tg_r. \qquad (16.18)$$

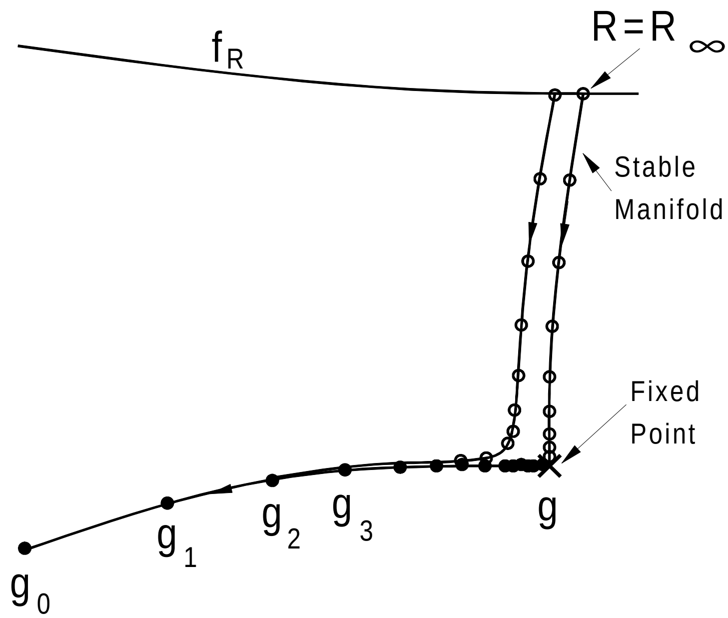

Figure 16.2: Flows of under T. The solid circles show successive g_r along the unstable direction given by operations of T. The empty circles show successive $T^n f_R$.

This defines the sequence of function $g_0, g_1 \ldots$ approaching the limit g. This is illustrated in figure 16.2.

Now thinking of these functions as one dimensional maps f, remember that if f has a stable 2^n cycle then $f(f(x))$ has a stable 2^{n-1} cycle, and since T gives just this functional composition (together with rescaling) the operation (16.18) acts to decrease the period order by a factor of 2. Thus we can choose to label the g_r such that g_r has a 2^r cycle (and then g itself shows a "2^∞" cycle). These g_r, defined only through referring to the fixed point structure, are universal versions of the sequence of maps studied in chapter 15. We have not fixed the normalization of $h(x)$, which affects the functional form of the g_r for small r. This choice determines what "type" of 2^r cycle g_r appears (superstable, marginally unstable etc.). The choice of normalization defined by the requirement $g_0(0) = 0$ selects the superstable cycles. (This is because the period 1 superstable cycle corresponds to the situation where the unit slope diagonal through the origin intersects the map curve at the maximum and this occurs when then maximum is at $y = 0$, see figure 16.1).

16.4 Bifurcations in the physical map

The fixed point in function space has one unstable direction: all other directions are contracting and we can construct a hypersurface in function space (the stable manifold) such that functions $f^{(\infty)}$ on this hypersurface evolve towards the fixed point g under the operation T. Since the stable manifold is codimension 1 we might expect a function $f_R(x)$ parameterized by a single parameter R to intersect the stable manifold for some value of R which we will call R_∞. Then under the operation of T

$$T^n f_{R_\infty} \to g \quad as \quad n \to \infty. \tag{16.19}$$

What about f_R for R near R_∞? Initially (for small n) $T^n f_R$ will follow $T^n f_{R_\infty}$, since $R - R_\infty$ is small, i.e. it will flow *towards* the fixed point. If $R - R_\infty$ is sufficiently small, $T^n f_R$ will approach very close to the fixed point for some range of n (figure 16.2). Since we understand the behavior here in terms of the fixed point and linearization we learn properties of the *physical* map from the properties of the fixed point and the *universal* maps g_r. However, since the fixed point has one *unstable* direction, eventually $T^n f_R$ will begin to flow away from the fixed point along a path close to the unstable direction $h(x)$, and we need to proceed carefully.

We split the n operations of T into a number of segments. First we operate some finite number q times, with q sufficiently large to bring $T^q f_R$ that the components along the unstable directions have decayed, i.e. to bring the function into the vicinity of the fixed point. The number q will not depend on $R - R_\infty$ for small values of this quantity, and is roughly the number of operations needed to bring f_{R_∞} into the vicinity of the fixed point. Taylor expansion then gives the amplitude along the unstable direction to be linear in the deviation $R - R_\infty$ for small enough $R - R_\infty$, so that

$$T^q f_R = g(x) + \bar{c}\,(R_\infty - R)\,h(x) \tag{16.20}$$

with $\bar{c}$ some number. The initial flow away from the fixed point is given by the linearization. So next we operate a number p times with p large (tending to ∞ as $R - R_\infty \to 0$), but small enough so that function remains in the linear regime along the unstable direction. This gives

$$\begin{aligned} T^{p+q} f_R &= T^p \left[g(x) + \bar{c}\,(R_\infty - R)\, h(x) \right] \\ &= g(x) + \bar{c}\,(R_\infty - R)\, \delta^p h(x) \end{aligned} . \qquad (16.21)$$

And then we complete T^n with a further $n-p-q$ actions, which may take the function into the nonlinear regime. Finally we may write

$$\begin{aligned} T^n f_R &= T^{n-p-q} \left[g(x) + \bar{c}\,(R_\infty - R)\, \delta^p h(x) \right] \\ &= T^{n-p} \left[g(x) + c\,(R_\infty - R)\, \delta^p h(x) \right] \end{aligned} \qquad (16.22)$$

with $c = \bar{c}\delta^q$.

Now if we make the special choices of $R = R_m$ defined by

$$c\,(R_\infty - R_m) = \delta^{-m} \qquad (16.23)$$

we have

$$T^n f_{R_m} = T^{n-p} \left[g(x) + \delta^{p-m} h(x) \right] = g_{m-n}, \qquad (16.24)$$

with g_{m-n} the universal superstable 2^{m-n}cycle. But each operation of T decreases the order of the cycle by a factor of 2, and so we see that f_{R_m} must have a superstable 2^mcycle.

Thus we have shown that if $R_\infty - R = c^{-1}\delta^{-m}$ then f_R will show a superstable 2^m cycle, and as m increases to infinity we will find a cascade of period doubling bifurcations, with superstable orbits at values of R with separation ratios given by the universal constant δ. We have shown the *existence of the period doubling cascade* with *universal* properties based on the existence of the fixed point of T with the assumed properties, and the assumption that f_R will cross the unstable manifold of the fixed point for some R. Since the stable manifold is of codimension 1 this latter should be a common occurrence.

16.5 Scaling of the map function

Putting in the scaling factors we have

$$f^{2^n}(x) = (-\alpha)^{-n}\, T^n f((-\alpha)^n x) \qquad (16.25)$$

for any function f. In particular using (16.24) for f_{R_m} we have

$$f_{R_m}^{2^n} = (-\alpha)^{-n} g_{m-n}((-\alpha)^n x) \qquad (16.26)$$

for large m, n with $m \geq n$.

In particular, for example, if we take $m = n$ then

$$\lim_{n\to\infty} f_{R_n}^{2^n} = (-\alpha)^{-n}\, g_0((-\alpha)^n x) \tag{16.27}$$

This says that the rescaled (by α^n) version of the 2^nth order functional composition of *any* map f (with a quadratic maximum) at the value of R giving a superstable 2^n cycle will tend to the universal function g_0 for large n. This operation was performed in chapter 15, demonstration 11. Similarly taking $n = m - 1$ will yield g_1, etc.

Alternatively if we first set $R = R_\infty$ then

$$\lim_{n\to\infty} f_{R_\infty}^{2^n} = (-\alpha)^{-n}\, g((-\alpha)^n x) \tag{16.28}$$

i.e. we approach the universal fixed point function, which we see to be the universal "onset of chaos" function. This limit was approached in chapter 15, demonstration 12.

16.6 Applications - the Lyapunov exponent

Suppose we want to calculate some property $P[f]$ of the map function f, such as the Lyapunov exponent. Since according to the development in the previous chapter

$$T^n f_{R_m} = g_{m-n} \tag{16.29}$$

for m, n large and $n \leq m$, if we can relate $P[f]$ to $P[Tf]$, then by repeated operation of T we can relate the desired property $P[f]$ to a property of the universal map i.e. to $P[g_n]$, which of course is universal. Using this approach we can show that certain properties of the physical map f are universal, and we can sometimes calculate the universal property precisely. Some of the calculations, such as for the Lyapunov exponent, are quite simple. Others, such as the power spectrum and the scaling of the separation of points in the orbit on which this depends, can be quite intricate.

The map $f = f_{R_m}$ has a stable 2^m cycle, and the Lyapunov exponent is given by the slopes of the map at the points x_i in the cycle:

$$\lambda[f] = \frac{1}{2^m} \sum_{i=0}^{2^m-1} \log\left|f'(x_i)\right| . \tag{16.30}$$

Writing this sum instead as the sum of successive pairs of points

$$\lambda[f] = \frac{1}{2^m} \sum_{j=0}^{2^{m-1}-1} \log\left(\left|f'(x_{2j})\right| \left|f'(x_{2j+1})\right|\right) \tag{16.31}$$

and using the chain rule for differentiating $f^2 = f(f(x))$ we can write this as

$$\lambda[f] = \frac{1}{2}\left\{\frac{1}{2^{m-1}} \sum_{j=0}^{2^{m-1}-1} \log\left(\left|f^2(x'_{2j})\right|\right)\right\} = \frac{1}{2}\lambda\left[f^2\right]. \tag{16.32}$$

It is easy to check that putting in the scaling factors in the definition of T does not change this result i.e.

$$\lambda[f] = \frac{1}{2}\lambda\left[f^2\right]. \tag{16.33}$$

This simply is the fact that we get the same divergence of orbits iterating f^2 half as many times.

Repeating this many times we have

$$\lambda\left[f_{R_m}\right] = \frac{1}{2^n}\lambda\left[T^n f_{R_m}\right] = \frac{1}{2^n}\lambda\left[g_{m-n}\right]. \tag{16.34}$$

Now we choose n to be comparable to m. Let us for example choose $n = m$, so that

$$\lambda\left[f_{R_m}\right] = \frac{1}{2^m}\lambda\left[g_0\right]. \tag{16.35}$$

But $\lambda[g_0]$ is some number independent of the nature of f and of the index m. This gives us the important scaling result $\lambda\left[f_{R_m}\right] \propto 2^{-m}$.

It is often convenient to rewrite the scaling result in terms of the evolution with the map parameter R (e.g. as in the Lyapunov plots of chapter 14). Thus we write

$$\begin{array}{rclcl} \lambda\left[f_{R_m}\right] & = & c\frac{1}{2^m} & = & ce^{-m\log 2} \\ R_\infty - R_m & = & c'\frac{1}{\delta^m} & = & c'e^{-m\log\delta} \end{array}. \tag{16.36}$$

Eliminating m we can write this as

$$\lambda \propto |R - R_\infty|^\beta \quad \text{with} \quad \beta = \frac{\log 2}{\log\delta} \simeq 0.45. \tag{16.37}$$

This functional dependence gives the shape of the envelope of the Lyapunov exponent for a fixed stability type (since in (16.36) this is what the values R_m refer to). The same result applies above R_∞, where positive Lyapunov exponents indicate chaotic dynamics (again for a fixed "type" of behavior, e.g. the "band merging" points–see chapter 17).

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