Physics 161: Homework 8

(February 23, 2000; due March 1)

Problems

1. **Pointwise Dimension**: Thie problem leads you through the proof showing that the "pointwise dimension" for any point $\vec{x}$ on the unstable manifold of a fixed point j of a two dimensional map F (with as nice properties as desired) is

$$D_p(\vec{x}) = 1 - \frac{\log \lambda_1^{(j)}}{\log \lambda_2^{(j)}}$$

   where $\lambda_1^{(j)} > 1$ and $\lambda_2^{(j)} < 1$ are the magnitudes of the unstable and stable eigenvalues of the Jacobean of F at the fixed point j. This is done by considering two small circles centered at $\vec{x}$ of radii l_1 and l_2 with $l_1/l_2 = 1/\lambda_2^{(j)}$. Now iterate the larger circle M times and the smaller circle $M+1$ times *backwards* towards the fixed point, and use the facts (i) measure is conserved by the mapping, and (ii) the measure is smooth along the unstable manifold of the fixed point, to compare the measures associated with the two circles at $\vec{x}$. (Draw some pictures!) This gives an estimate for the scaling of the measure with box size at $\vec{x}$ and hence $D_p(\vec{x})$.

   The generalization to unstable limit cycles is straightforward by considering F^n. The result suggests that although the "typical" point will give a pointwise dimension equal to the information dimension D_1 (c.f. chapter 9) there will be significant fluctuations in numerical estimates.

2. For the Hénon map with $a = 1.4$, $b = 0.3$ there are two fixed points, one of them on the attractor.

   (a) Find the fixed points.

   (b) Identify which fixed point is on the attractor by comparing with a numerical simulation of the attractor.

   (c) For the fixed point on the attractor compute the eigenvalues and eigenvectors of the Jacobean of the map at the fixed point.

   (d) Compare the directions of the eigenvectors with the structure of the attractor near the fixed point.

   (e) Calculate the pointwise dimension at this fixed point according to the expression in problem (1), and compare with various D_q estimated numerically for the Hénon map.

   (f) We could imagine constructing the stable manifold by iterating the *inverse* of the Hénon map starting from a range of initial points displaced from the fixed point by very small distances along the stable eigenvector. Construct the inverse map to allow us to do this.

   For your convenience a copy of the Hénon attractor is shown in figure 1. Note if you use the *2dmap* applet to construct your own plot that the axis ranges are scaled so that the range *x_min* to *xmax* is plotted as 0 to 1, and similarly for the y-axis.

   Plots of the stable manifold make pretty pictures, for example Aligood et al. [1] plates 24 and 25. These plots also indicate that the Hénon attractor is not hyperbolic, because they show

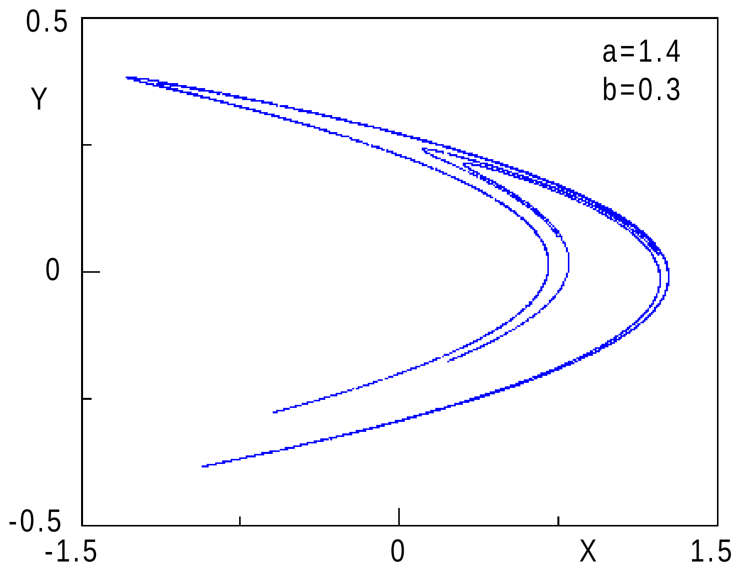

Figure 1: Henon attractor at $a = 1.4$, $b = 0.3$

that the stable manifold is tangent to the attractor at some points. At these points the unstable tangent vector (always parallel to the attractor) and the stable tangent vector are parallel, and so do not span the space as required for hyperbolicity. The numerical scheme for constructing these plots is a little subtle—it is discussed in Aligood et al. (example 10.5), and also in §9.1 of *Dynamics* by Nusse and Yorke [2].

References

[1] K.T. Aligood, T.D. Sauer, and J.A. Yorke, *Chaos: an Introduction to Dynamical Systems* (Springer, New York 1996).

[2] H.E. Nusse and J.A. Yorke, *Dynamics: Numerical Explorations*, 2nd edition (Springer, New York 1998).