Physics 161: Homework 5

(February 2, 2000; due February 9)

Problems

1. **More on Dimensions**: For $l_1 = l_2 = l$ in the "two scale factor Cantor set" construction, corresponding to $\lambda_a = \lambda_b = \lambda$ in the bakers' map, direct box-counting calculations become easier, since all the elements at the nth level of construction have the *same* length. (Of course the measures still vary if $p_1 \neq p_2$.) This simplification allows us to investigate some of the other "dimensions" introduced in chapter 9. (As a hidden bonus, the "MyFunction" choice in the *1dmap* applet plots an fixed-y section of the bakers' map for this case with the parameter a giving α and the parameter b giving $\lambda_a = \lambda_b$, so you can use this to look at other properties of the attractor, such as the singular measure, numerically.) It is easy then to evaluate the information D_1 directly by box counting, and to study the pointwise dimension and θ-capacity.

   I was going to set this as a problem, but decided to add the discussion to the notes. So read the new section on "Other Dimensions" in chapter 9 instead!

2. **Experiment:** Find a report in a journal, magazine or book of an experiment showing chaotic dynamics in a system that was *not* mentioned in the notes or class. Give a reference, and briefly describe the system and measurements.

3. **Bifurcations:** Consider the bifurcations of the stationary solutions of a particle undergoing damped one dimensional motion in a potential $V(x)$ described by the equation of motion

$$\eta \dot{x} = -\frac{dV}{dx} \tag{1}$$

   with η a damping constant.

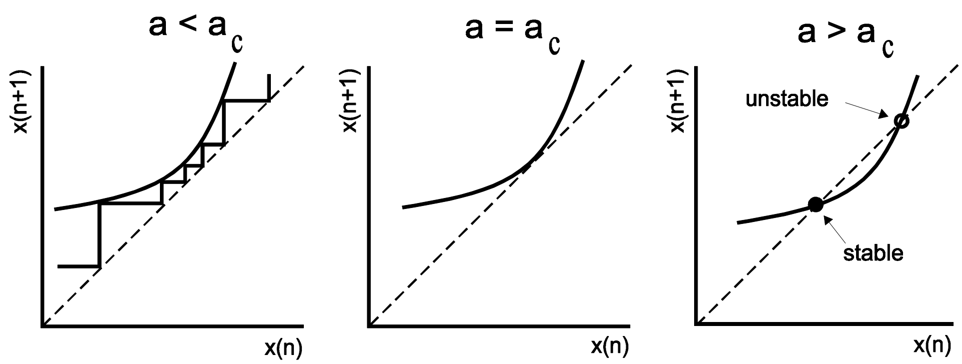
   (a) For each of the two cases below: study how the stationary solutions vary as the parameter r passes through zero; explicitly evaluate the stability of the solutions by performing a linear stability analysis about each one; identify the type of bifurcation from the $x = 0$ solution; by judiciously rescaling the variables explicitly reduce the equation of motion for small x and r to the appropriate "normal form"; and (iv) sketch the potential $V(x)$ for r small and negative, r zero, and r small and positive, and relate the stable and unstable solutions to the form of the potential.

      i. the potential $V(x) = -rx^2 + bx^4$ with b positive;

      ii. the potential $V(x) = -rx^2 + bx^3 + cx^4$ with b and c positive;

   (b) By qualitatively sketching the form of the potential $V(x) = -rx^2 - bx^4 + cx^6$ for b small and positive, c positive, and as r varies, discuss the bifurcations that occur. Sketch the solution you would expect to see as r is increased and decreased over a wide range.

4. **Type I Intermittency:** An interesting way that chaos can appear or disappear is through "intermittency" where the dynamics gets trapped in the vicinity of a simple non-chaotic dynamics for a time that diverges to infinity approaching the transition point. The simplest type of intermittency (known as type I) is illustrated by a "tangency saddle-node bifurcation" in a one dimensional return map. Consider the following situation that might occur as a is increased in the quadratic map:



For $a > a_c$ there is a stable fixed point (as well as an unstable one). For $a < a_c$ there are no fixed points here, and the trajectory moves away to explore distant regions, which we will assume to be chaotic. Locally the map can be described as

$$y_{n+1} = -\epsilon + y_n + y_n^2$$

where ϵ is proportional to $a - a_c$ and a $y_n = x_n - x_s$ with x_s a constant equal to the value of x exactly at the saddle node.

(a) Verify that this equation reproduces the saddle node bifurcation sketched in the figure.

(b) For small and negative ϵ the behavior near $y = 0$:

$$y_{n+1} - y_n = -\epsilon + y_n^2$$

can be replaced by a continuum equation

$$\frac{dy}{dt} = -\epsilon + y^2$$

where $t = n$ gives the discrete mappings. By solving this equation show that the number of iterations it takes for x_n to get through the bottleneck scales as $|\epsilon|^{-1/2}$. As $|\epsilon| \to 0$ we would expect to see "laminar" regions (i.e. when $y \simeq 0$) of duration increasing as $|\epsilon|^{-1/2}$.

It turns out that the appearance of the period 3 orbit out of the chaotic motion as a is increased above about 3.83 in the quadratic map occurs via type 1 intermittency. You can first study this transition using the *bifurcation* option in the program *1dmap*—describe what you see. Then use the *time series* option (or, of course, your own program) for the following:

(a) Study the behavior for $a = 3.8282$. I found the behavior: Period 3 (70 iterations); Chaotic (12 iterations); Period 3 (68 iterations); Chaotic (60 iterations); Period 3 . . . ; where "Period 3" means the orbit is close to repeating after 3 iterations and "Chaotic" means the orbit appears complex.

(b) Since we are close to a period 3 orbit look at $f^3(x) = f(f(f(x)))$. Repeat the iterations and observe the trapping of the orbit near the almost tangency. (In *1dmap* you can expand this region using the mouse providing you choose an initial value in the range you are interested in.)

(c) Study the dependence of the "period 3" residence time on a.

Type I intermittency occurs at a saddle node bifurcation. There are other types of intermittency, with different power law dependences of the laminar residence times: type II intermittency may occur near a subcritical Hopf bifurcation and type III intermittency near an subcritical period doubling bifurcation.