Physics 161b: Homework 1

Further Reading:

The original paper by Lorenz (available in the folder on reserve in Millikan) is very readable. Lesson 1 also mentions some physical systems for which the Lorenz model is an *accurate* description, and gives references to the original papers.

Problems:

1. **The Lorenz Model**

   The Lorenz model was derived as a description of the fluid circulation in convection "rolls". The X variable in my notation is proportional to the circulation velocity. For the "standard" parameters $r = 28$, $b = 8/3$, $\sigma = 10$ (a, b, c in the notation of the demonstrations):

   (a) Show that there are *three* time independent solutions or "fixed points" (given by setting $\dot{X} = \dot{Y} = \dot{Z} = 0$).

   (b) By starting with initial values of X, Y, Z near (but not exactly at) each of these fixed points in turn, use the demonstrations or other numerical methods to study the time evolution of X, Y, Z. Describe *in words* the behavior of the circulation velocity in the dynamics you observe.

2. **The Duffing Oscillator**

   A modification of the pendulum equation is the driven (inverted) Duffing equation

   $$\ddot{x} + \gamma \dot{x} - x + x^3 = g \cos(\omega_D t)$$

   which corresponds to a particle moving with damping γ in a one dimensional double well potential $V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 + \frac{1}{4}$ with a sinusoidal driving of strength g and frequency ω_D. (You can look at Moon and Holmes, J. Sound Vibration **65**, 275 and **69**, 339 (1979), or Moon's book for an experimental implementation—the "Moonbeam(!)".)

   *The first three parts below should be done **before** you use the Duffing demonstration program.*

   (a) Sketch qualitatively the orbits in the two dimensional phase space (x,$v = \dot{x}$) in the *undamped*, *undriven* case ($\gamma = g = 0$). Consider in particular orbits with energy $E = \frac{1}{2}\dot{x}^2 + V(x)$ given by (a) $E < \frac{1}{4}$, (b) $E = \frac{1}{4}$, and (c) $E > \frac{1}{4}$. Describe the nature of the orbits.

   (b) Sketch what you anticipate for some orbits for small damping, but still no driving $g = 0$, identifying stable and unstable fixed points.

   (c) How do phase space volumes evolve in time with nonzero driving *and* dissipation?

   Now investigate the behavior numerically - you can use the ODE applet and choose the Duffing model (the translation of notation is $g \to a$, $\gamma \to b$, and $\omega_D \to c$).

(d) First look at the case for no damping and no driving, $g = \gamma = 0$, and look at different orbits, e.g. starting new trajectories by clicking on the running plot of the demonstration. Do your results agree with what you expected?

(e) Now turn on a small damping e.g. $\gamma = 0.05$. Again look at trajectories from different initial conditions. From your observations, what are the attractors (regions of phase space to the trajectories converge at long time)? Sketch their basins of attraction in the x, v phase space (i.e. the initial conditions that are attracted to the various attractors).

(f) Investigate numerically the behavior with both driving and damping. Use the parameters $g = 0.3$, $\omega_D = 1$, and first $\gamma = 0.22$, and then $\gamma = 0.25$ (and others if you wish). Be careful to distinguish between transient behavior, and long time behavior (the attractor). Use the phase-space projections, time series, the power spectrum and Poincaré sections as appropriate to characterize the behavior.