Lecture 8 Supplementary Notes: Amplitude Equations

Michael Cross, June 1, 2006

These are more detailed notes on the derivation and properties of the one-dimensional amplitude equation.

Introduction

It is easy enough to discuss the evolution of infinitesimal perturbations of a uniform state into saturated, stationary, spatially periodic solutions. By restricting attention to these simple solutions, it is straightforward to formulate the effects of the nonlinearities, using analytical methods near threshold (bifurcation theory) and fairly simple numerical methods further from threshold. However, most realistic geometries do not permit spatially periodic solutions since these are usually not compatible with the boundary conditions at the lateral walls. Even if periodic solutions are consistent with some finite domain, they do not exhaust all the possibilities. More typically, only over small regions do patterns have the ideal form (stripes, hexagons, etc.), and these ideal forms are distorted over long length scales, and disrupted in localized regions by defects. In addition the distortions and defects are often time-dependent. The **amplitude equation** formalism provides a method to study spatial distortions of ideal patterns and their time dependence.

Amplitude equations capture three basic ingredients of pattern formation: the growth of the perturbation about the spatially uniform state, the saturation of the growth by nonlinearity. and what we will loosely call **dispersion**, namely the effect of spatial distortions. The interplay of these three effects lies at the heart of pattern formation, and amplitude equations have yielded many useful quantitative insights. In addition, amplitude equations provide a natural extension of the classification of pattern forming systems based on the type of linear instability into the weakly nonlinear regime. Such behavior, common to a class of diverse systems, is often called **universal**[1]. The remarkably similar pattern formation that is observed in diverse systems can often be understood as a consequence of the universal forms of the amplitude equations.

The increased generality of the types of states that can be investigated within the amplitude equation comes with the penalty that amplitude equations have a restricted range of validity. The amplitude equation formalism is derived as an expansion about threshold, and so the quantitative applicability is restricted to small values of the expansion parameter

$$\varepsilon = \frac{p - p_{\mathrm{c}}}{p_{\mathrm{c}}}, \tag{1}$$

where p is the control parameter such as the Rayleigh number R and p_{c} is its critical value above which the uniform state becomes unstable in the ideal infinite system. In addition, the distortions that can be studied are only those modulations of ideal patterns (stripes, squares, hexagons, etc.) that vary slowly in space and time compared to the basic length and time scales of the dynamical equations. In addition, a slow variation of the pattern that leads to large reorientations over large distances such as may occur in a rotationally invariant system is not contained within the amplitude equations that have been derived at the time of writing.

The complex amplitude that describes modulations of a stripe state near threshold is introduced in §**??**. The equation of motion satisfied by this amplitude is derived using symmetry arguments, with the parameters fixed by referring to a number of simpler calculations. (A systematic but more technical derivation using the method of multiple-scale perturbation theory is given in separate notes) The amplitude equation is a pde, and its solution requires the knowledge of boundary conditions, which are derived from the boundary conditions

[1] The word *universal* is not meant to imply that the behavior applies to every system, but rather to a whole class of systems characterized by broad similarities, such as symmetries and instability type. The behavior for systems in different "universality classes" may be totally different.

on the physical fields. In the common case of boundaries that tend to suppress the pattern, we show that the boundary conditions take on a simple form that is also universal. A more complex analysis addresses the appropriate boundary conditions where the boundaries strongly drive the pattern.

We then discuss important general properties of the amplitude equation. We first display the universality of the amplitude equation and discuss some physical implications of universality. The simplest way of showing the universality is to note that by choosing new length, time, and magnitude scales, the equation can be reduced to a parameter independent form. After this, we show that in many situations the dynamics of the one dimensional amplitude equation has the special property of being "potential" (also called "relaxational"). This means that we can find a functional (called the potential and sometimes the Lyapunov functional) that decreases monotonically over time for any initial condition of the amplitude. The existence of this potential is a mixed blessing. On the one hand, the potential often allows a more intuitive understanding of the dynamics and can greatly simplify various calculations. On the other hand, since this feature is reminiscent of the systematic increase or decrease of thermodynamic potentials approaching thermodynamic equilibrium, and because the existence of the potential turns out to be an artifact caused by retaining only the lowest-order nonzero terms in the perturbative expansion that yields the amplitude equation, we may worry that the potential nature of the dynamics indicates the failure of the amplitude equation to encompass the full richness of possible behaviors.

We conclude with a discussion of three simple but important applications: the effect of boundaries on the nonlinear pattern; the stability balloon near threshold; and the slow dynamics of special distortions corresponding to slowly varying compression or dilation of the pattern which is captured by studying the phase dynamics, i.e. how the phase of the complex amplitude itself evolves in time.

Origin and Meaning of the Amplitude

We define a spatially dependent complex amplitude $A(x,t)$ in terms of perturbations $\mathbf{u}_p = \mathbf{u}(x,z,t) - \mathbf{u}_b(z)$ from the uniform base state $\mathbf{u}_b$ by the equation

$$\mathbf{u}_p(\mathbf{x},z,t) = \left[A(x,t)\mathbf{u}_c(\mathbf{x}_\parallel)e^{iq_c x} + \text{c.c.}\right] + \text{h.o.t.}, \tag{2}$$

where, as before, "c.c." denotes the complex conjugate of the prior expression and "h.o.t." means "higher order terms" that are smaller in magnitude than the displayed terms in the limit that the reduced bifurcation parameter Eq. (1) becomes sufficiently small. The ansatz Eq. (2) is key to the further development and warrants careful discussion.

We choose to base our expansion around the critical onset mode $\mathbf{u}_c(\mathbf{x}_\parallel)e^{iq_c x}$ where q_c is the critical wave number that minimizes the onset control parameter p_c. The function $\mathbf{u}_c(\mathbf{x}_\parallel)$ is the shape of the critical unstable mode and is known to us from the linear stability calculation. We assume first a situation where the modulations of the critical onset mode only depend on the direction normal to the stripes.

The amplitude $A(x,t)$ introduces a slow *modulation* of the critical solution in the extended direction x. Since we are basing the expansion on the behavior of the sinusoidal mode at q_c, we restrict the spatial variation of A to be on a much longer length scale than the basic length scale q_c^{-1} of the pattern formation. With this constraint, only wave numbers q close to zero will appear in the Fourier representation of A, and correspondingly only wave numbers close to q_c will appear in the representation of the perturbation $\mathbf{u}_p$[2].

[2]The ansatz Eq. (2) is basically a generalization of the phenomenon of *beating* to include many modes that all vary with nearly the same wavenumber. You have perhaps heard the result of playing two sinusoidal sound waves of slightly different frequencies, which combine to produce what sounds like a single tone whose intensity varies slowly with time. Mathematically, the expression $\sin(\omega_1 t) + \sin(\omega_2 t) = [2\cos((\omega_2 - \omega_1)t)]\sin((1/2)(\omega_1 + \omega_2)t)$. For two frequencies ω_1 and ω_2 that are close in value. this expression corresponds to a *slow* periodic modulation $A(t)\sin(\omega_a t)$ of a signal at their average frequency $\omega_a = (\omega_1 + \omega_2)/2 \approx \omega_1 \approx \omega_2$, where the amplitude $A = 2\cos(\Delta\omega t)$ varies slowly specifically because the range of frequencies $\Delta\omega = \omega_2 - \omega_1$ about the average is small.

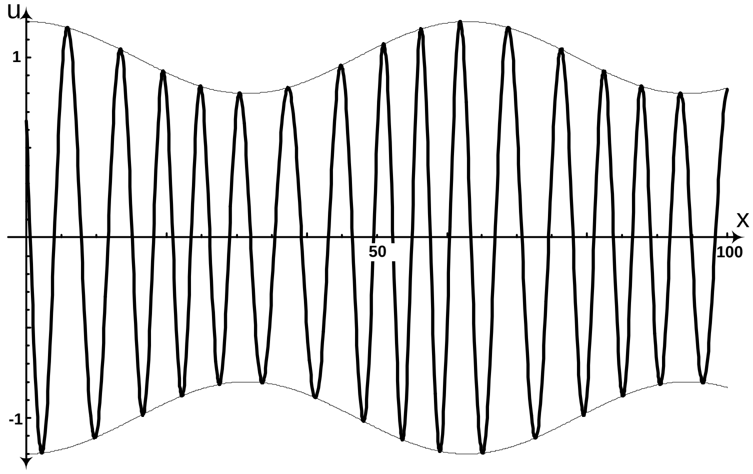

Figure 1: An example of a modulated sinusoidal solution. The function plotted (heavy curve) is $u = A(x)e^{iq_c x} + \text{c.c.}$ with $q_c = 1$ and $A(x) = (0.5 + 0.1\cos(0.1x))e^{i\cos(0.2x)}$. The magnitude modulation is shown by the two light curves $\pm|A(x)|$. The phase modulation gives a varying periodicity to the heavy curve. (The numerical values appearing in $A(x)$ are chosen arbitrarily, within the constraint that the variation of $A(x)$ should be slow compared to the unmodulated wave, for illustration purposes.)

Restricting A to vary slowly over space necessarily implies that A varies slowly over time. The reason is that, at the linear level, an onset mode with wave vector q has an exponential growth rate σ_q that goes to zero as onset is approached ($\varepsilon \to 0^+$ and $q \to q_c$). The function $\mathbf{u}_p$ then consists of a superposition of modes whose growth rates are all close to zero.

A simple illustration of a one-dimensional modulated periodic state $u(x) = A(x)e^{iq_c x} + \text{c.c.}$ is shown in Fig. (1). We use a base wave number $q_{\rm c} = 1$, and an illustrative modulation function

$$A(x) = [0.5 + 0.1\cos(0.1x)]e^{i\cos(0.2x)}, \tag{3}$$

The expression $|(1/A)\, dA/dx|$ is the effective local wave number in the Fourier expansion of A and you can verify graphically that its maximum is about 0.2. Since this is small compared to $q_{\rm c} = 1$, we can indeed think of A as a slow modulation of the periodic state $\cos x$. An obvious feature of Fig. 1 is the modulation of the magnitude of the sinusoid but if you look carefully, you will also see that the local periodicity of u (e.g., the distance between two adjacent zero crossings) is no longer constant, so that the wave number also has a slow spatial modulation.

Our discussion so far of the "slow" spatial and temporal variations of the amplitude A compared to the "fast" dynamics of the physical fields has been informal and so somewhat vague. The mathematical formalism of multiple scales makes the discussion of slow scales precise and allows the higher terms "h.o.t." in Eq. (2) to be calculated systematically.

It is often useful to express the amplitude $A(x, t)$ in magnitude-phase form

$$A = ae^{i\phi}, \tag{4}$$

where $a(x, t)$ is its real-valued magnitude and $\phi(x, t)$ is its real-valued phase. The magnitude and phase then play different roles in the dynamics of A. The magnitude gives the size of the perturbation $\mathbf{u}_p$ near onset and typically evolves relatively quickly, often showing an exponential decay to a steady value. On the other hand, the phase sets the *position* of the growing stripes, e.g., a constant phase ϕ_0 translates the field $\mathbf{u}_p$ rigidly by a distance $-q_c^{-1}\phi_0$ in the x direction. Because of its link to translational and rotational symmetries of the system, the phase generally evolves more slowly than the magnitude and its dynamics can often be isolated and studied separately.

A slow variation in the amplitude's phase corresponds to a stretching of the wave number of the critical state (see Fig. 1). We can see this by examining the effect of the lowest-order non-constant terms of a Taylor-expansion of the phase $\phi = Q_x x + \cdots$ in the vicinity of some point (which we can assume to be the origin by translational invariance) and by assuming that $|Q_x| \ll q_c$ so that the phase is slowly varying (large changes in x lead to only small changes in ϕ). The linear variation with x

$$\phi = Q_x x \tag{5}$$

corresponds to a change in the wave number of the pattern so that the wave vector of the perturbation Eq. (2) is q with

$$q = q_c + Q_x. \tag{6}$$

We conclude our discussion of the ansatz Eq. (2) with a few comments about where the higher-order-terms "h.o.t." come from. If the expansion in ε of Eq. (1) is formally carried out, corrections indeed arise that are proportional to higher and higher powers of ε. Some of the corrections come from spatial harmonics that are generated by the nonlinearities, for example cubing the sinusoid $\cos qx$ creates a harmonic $\cos 3qx$. But there are also corrections that arise at the linear level since, for a spatially varying amplitude, the structure $\mathbf{u}_c(\mathbf{x}_\parallel)$ will not give the precise solution to the evolution equations. For example, a variation corresponding to a shift of wave number will change $\mathbf{u}_c$ to $\mathbf{u}_q$ in the exponentially growing solution. In addition, the mode structure is perturbed if the control parameter is not exactly equal to its threshold value, which also leads to higher-order-terms in Eq. (2).

Derivation of the Amplitude Equation

Phenomenological Derivation

The evolution equation for the amplitude A known as the *amplitude equation* can be derived by substituting Eq. (2) into the evolution equations for the physical field $\mathbf{u}$ and then using the formal multiple scales expansion technique. Instead, we will proceed more phenomenologically to deduce directly the form of the amplitude equation. This involves writing down terms that are low order in the various small quantities and then considering how various symmetries restrict the possible form. While this phenomenological approach suffices for the simple case of the lowest-order one dimensional amplitude equation, ultimately a formal expansion is needed to understand the regime of validity of the amplitude equation, to obtain higher-order corrections that may important for understanding particular experiments, and to extend the method to more complicated situations such as degenerate bifurcations.

We argue that the one dimensional amplitude equation for a modulated stripe state near the onset instability takes the form

$$\tau_0 \partial_t A(x, y, t) = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \tag{7}$$

where ε is the reduced bifurcation parameter Eq. (1). The quantities τ_0, ξ_0, and g_0 are constants that depend on details of the physical system and can be calculated from the known evolution equations. In contrast, the mathematical form of Eq. (7) does *not* depend on details of the physical system undergoing a type I-s transition. Its form is dictated completely by symmetry arguments, by a smoothness assumption that constrains which derivatives can appear, and by the fact that we are expanding about a base solution that minimizes the onset control parameter.

The symmetry requirements that constrain the possible form of an amplitude equation arise from the need for Eq. (7) to be consistent with the symmetries that leave invariant the evolution equations for the physical field $\mathbf{u}$, with the correspondence given by Eq. (2). Thus we require that Eq. (7) be invariant under:

1. the substitution $A \to Ae^{i\Delta}$ with Δ a constant, which corresponds to a translation of the pattern $\mathbf{u}_p$ through a distance $-\Delta/q_c$ in the x-direction;

2. the double substitution $A \to A^*$ followed by $x \to -x$, which corresponds to inversion of the horizontal coordinates (parity symmetry);

As an example, under the substitution $A \to Ae^{i\Delta}$ for some constant Δ, the solution $\mathbf{u}_p$ in Eq. (2) becomes

$$\mathbf{u}_p(\mathbf{x}_\perp, z, t) = \left[Ae^{i\Delta}\mathbf{u}_c(\mathbf{x}_\parallel)e^{iq_c x} + \text{c.c.}\right] + \text{h.o.t.} \tag{8}$$

$$= \left[A\mathbf{u}_c(\mathbf{x}_\parallel)e^{iq_c(x+\Delta/q_c)} + \text{c.c.}\right] + \text{h.o.t.}\,, \tag{9}$$

which indeed corresponds to a translation of the field $\mathbf{u}_p$ by the amount $-(\Delta/q_c)$.

The required invariance of the amplitude equation under the symmetries of translation and parity restricts the possible terms in the amplitude equation in the following ways. First, we observe that algebraic products of A and of its complex conjugate A^* that lead to odd powers such as A, $|A|^2A$ $(= A^*A^2)$, $|A|^4A$, and so on are invariant under all the symmetries and so can appear in the amplitude equation. Invariance under the substitution $A \to Ae^{i\Delta}$ rules out even powers such as A^2, $(A^*)^2$, $|A|^2$, and $|A|^2A^2$ as well as some odd powers such as A^3, $(A^*)^3$, and $|A|^2A^3$. The terms A and $|A|^2A$ are the simplest ones that lead to growth and saturation and so appear in Eq. (7). Although $|A|^2A$ is higher order than the linear term, the coefficient of the linear term is small near onset, which corresponds precisely to the vanishing of this coefficient. (A subcritical transition would require also the next allowed term $|A|^4A$.) We will discuss in a moment why we do not include nonlinear terms such as $|A|^2\partial_x^2 A$ that are allowed by symmetry but that contain partial derivatives.

Let us next consider what kinds of derivatives of A can appear in the amplitude equation. There must be some kind of time derivative since this is an evolution equation and the simplest guess would be that a first-order derivative $\partial_t A$ is sufficient. This is allowed by the above symmetries but is also the simplest choice consistent with a symmetry not mentioned above but implicit in all driven-dissipative pattern-forming systems, that the dynamics is *not* invariant under the time-reversal symmetry $t \to -t$ so that there is a preferred direction of the dynamics over time. The absence of time-reversal symmetry rules out the possibility that the time derivative term is an even power such as ∂_t^2, which is otherwise allowed by the space-related symmetries. Thus we have justified the occurrence of the first-order derivative on the left side of Eq. (7).

We next observe that a first-order spatial derivative of the form $i\partial_x A$ is allowed by the above symmetries, for example it is consistent with the parity symmetry $A(x) \to A^*(-x)$. However, such a term can be eliminated by a redefinition $A \to \bar{A}e^{i\Delta x}$ for a suitable constant Δ and so would play no essential role in the dynamics. Such a change in fact corresponds to a change of the reference wave number: the choice of the critical wave number q_c as the reference—the wave number that minimizes the onset control parameter p_c—already implies the absence of the $i\partial_x A$ term. We therefore assume that no such term appears in the amplitude equation.

A second order derivative term $\partial_x^2 A$ is consistent with all the symmetries, and will occur in the amplitude equation. For an amplitude $A(x,t)$ describing slow modulations, higher order derivatives will be correspondingly smaller (roughly by the ratio of the length scale of the modulation to the basic wavelength of the pattern). We will therefore truncate the expansion at second order in the derivatives. For the same reason, we ignore nonlinear terms with spatial derivatives such as $|A|^2\partial_x A$ since such a term is smaller than the existing cubic term $|A|^2A$[3].

[3]Note that we are assuming that the amplitude equation is “smooth” so that we may expand in successive integral order derivatives

Deduction of the Parameters

Once we have accepted the form of the amplitude equation Eq. (7), the unknown parameters τ_0, ξ_0, and g_0 can be deduced from simpler calculations. Thus if we consider a small amplitude disturbance

$$A = \delta A(t)e^{ikx}. \tag{10}$$

and linearize the amplitude equation about the zero solution $A = 0$, we see that the time dependence is exponential with a growth rate $\tau_0^{-1}(\varepsilon - \xi_0^2 k^2)$. But by the correspondence Eq. (2), this is the growth rate of a small physical perturbation $\mathbf{u}_p$ at wave vector $q = q_c + k$ and so must correspond to the growth rate $\sigma(q)$ of the linear stability analysis for the uniform base state $\mathbf{u}_b$. Thus we have

$$\sigma(q) = \tau_0^{-1}\left[\varepsilon - \xi_0^2(q - q_c)^2\right] + \cdots , \tag{11}$$

for small ε and small $q - q_c$. The parameters τ_0 and ξ_0 can be read off from the growth rate $\sigma(q)$ calculated from the linear instability analysis of the full evolution equations. Alternatively, we can split the calculation into two pieces: first compare the amplitude growth rate with the dependence on ε of the growth rate σ_q at the critical wave number

$$\sigma(q_c) = \tau_0^{-1}\varepsilon + \cdots , \tag{12}$$

and then compare with the dependence of the critical control parameter value on wave numbers near q_c

$$\varepsilon_c(q) = \xi_0^2(q - q_c)^2 + \cdots . \tag{13}$$

The coefficient g_0 determines the saturation amplitude of the critical mode

$$|A| \to (\varepsilon/g_0)^{1/2}, \tag{14}$$

and so g_0 can be found from a calculation for the nonlinear saturation of the critical mode without the complications of spatial modulations.

Although the constants τ_0, ξ_0, and g_0 are needed to compare predictions of the amplitude equation with experiments, the qualitative dynamical behavior of the solutions to Eq. (7) does not depend on their values. We can see this by rescaling the variables in Eq. (7) as follows

$$\tilde{A} = g_0^{1/2}A, \qquad \tilde{x} = x/\xi_0, \qquad \tilde{t} = t/\tau_0, \tag{15}$$

to obtain an equation in which only the parameter ε remains

$$\partial_{\tilde{t}}\tilde{A} = \varepsilon\tilde{A} + \partial_{\tilde{x}}^2\tilde{A} - |\tilde{A}|^2\tilde{A}. \tag{16}$$

Solutions of Eq. (16) can be compared with experiment by transforming back to the "physical" variables using Eq. (15). From the scaling, we see that τ_0, ξ_0, and g_0 serve to set the time, length, and magnitude scales for the problem.

$\partial_x^n A$. Rather surprisingly, the "obvious" assumption of smoothness does not hold near onset for some pattern-forming systems. An example is Rayleigh-Bénard convection between free-slip plates, although the difficulties only appear in the two-dimensional amplitude equation. In this example something called a horizontal mean flow appears that depends nonlocally on the physical fields. The lowest-order amplitude equation then turns out to involve two coupled fields whose dynamics can not be reduced to a single amplitude equation with simple derivative terms.

Method of Multiple Scales

The method of multiple scales formalizes the expansion procedure about threshold by tying together the various small effects: the small distance from threshold ε, the slow time dependence, the weak nonlinearity represented through saturation at a small magnitude $|A|$, and the slow spatial modulation. The derivation is quite technical, and for the lowest order one dimensional amplitude equation only serves to justify the phenomenological derivation just given. The approach is, however, widely used in the theory of pattern formation, and provides an introduction to a widely used type of perturbation scheme that is not typically encountered in a standard physics education (although students of other disciplines, such as Applied Math, are likely to be familiar with the method). In more complicated situations it may not be possible to derive the amplitude equation by symmetry arguments and matching to simpler calculations, and in these cases the method of multiple scales becomes necessary.

The phenomenological approach is usually inadequate if we need to extend the calculation to higher order in the expansion in ε, because there are then too many terms to be pinned down by simple arguments. The extension to higher order may be necessary not just for quantitative accuracy, but because the results from the lowest order calculation may sometimes be qualitatively misleading. An example of this is nature of the nonlinear states in a finite one dimensional system with realistic boundaries. Here the lowest order amplitude equation suggests that a continuum of states exist, corresponding to an arbitrary translation of the stripes relative to the ends, (see §6). It is only if the calculation is extended to the next order that the correct result is recovered, namely a discrete set of states where the stripes have a preferred position relative to the ends.

Boundary Conditions

To solve a pde such as the amplitude equation, Eq. (7) or Eq. (16), boundary conditions must be specified. A simple case, often used in simple theoretical analyses, would be periodic boundary conditions over some domain. To make contact with experiment we must use more realistic boundary conditions.

If the boundaries (taken to be at $\pm l/2$in our one dimensional system) tend to inhibit the onset of the pattern the boundary conditions for Eq. (7) take the form[4]

$$A(x = \pm l/2) = 0. \tag{17a}$$

A stationary rigid wall often has this effect on fluid patterns, since the motion of the fluid is quenched by viscous coupling to the wall. Note that these boundary conditions are again universal, independent of much of the underlying physics leading to the pattern formation.

On the other hand the boundaries may serve to drive the pattern formation. An example is a heated wire around the sidewall of a Rayleigh-Bénard convection system, which will drive convection currents at all Rayleigh numbers, even below the threshold of the instability in the ideal infinite system. Below the ideal threshold, the convection currents will be confined to a narrow region near the walls. As threshold is approached, the convecting region will expand, and will fill the system above threshold. Since there is no sharp onset of convection in such a system, the bifurcation is said to be imperfect. Similarly in the Taylor-Couette system, a rigid end wall will tend to drive a circulating vortex at the ends (called the Ekman vortex) even below the onset of rolls in the ideal system. It will usually be the case that the local driving by the ends is not small compared with the expansion parameter ε, since the behavior near the ends is quite different than in the bulk, and the value p_c of the control parameter should not play any special role here. It can then be

[4] Remember that the magnitude of the amplitude is expected to scale as $\varepsilon^{1/2}$ near threshold. Equation (17a) should be interpreted in terms of the amplitude going to zero on this scale. There may be $O(\varepsilon)$ corrections to the zero on the right hand side.

argued that the boundary conditions must take the form

$$A(x \to \pm l/2) = \frac{C_\pm}{|x \mp l/2|}. \tag{18}$$

Here the complex constants $C_\pm$ will depend on the details of the boundary effects, and the values would have to be calculated by matching to a more complete solution of the strongly driven region near the ends. The divergence of $|A|$ approaching the boundary corresponds to the physical statement that the disturbance becomes large near the end. Of course the amplitude equation description breaks down very close to the end (remember that the modulations given by the amplitude must be slowly varying), so there is no actual divergence of physical quantities.

The arguments leading to these boundary conditions are rather delicate, and you might prefer to accept them as resulting from matching to the full solution in the end regions. The details of such a calculation however go beyond the level of these notes.

Properties of the Amplitude Equation

Universality and Scales

In our discussion of Eq. (16) above, we found that we could eliminate the scale factors τ_0, ξ_0, and g_0 from the amplitude equation by transforming time, space, and magnitude variables. We can modify this transformation of variables as follows:

$$\bar{A} = |\frac{g_0}{\varepsilon}|^{1/2} A, \qquad X = \frac{|\varepsilon|^{1/2}}{\xi_0} x, \qquad T = \frac{\varepsilon}{\tau_0} T, \tag{19}$$

to obtain the *fully scaled amplitude equation* from which *all* the parameters have been removed

$$\partial_T \bar{A} = \pm \bar{A} + \partial_X^2 \bar{A} - |\bar{A}|^2 \bar{A}. \tag{20}$$

(The positive sign for the first term on the right hand side corresponds to above threshold $\varepsilon > 0$, and the negative sign to below threshold $\varepsilon < 0$). In this equation there are no parameters that depend on the physical nature of the system. This dramatically demonstrates the *universality* of pattern forming phenomena near onset when the amplitude equation is a good description, since we can analyze the behavior of Eq. (20) without referring back to the physical nature of the system. This shows us that all one dimensional stripe states near threshold will have the *same* properties.[5]

The *absence* of any explicit dependence on the small parameter ε in the scaled amplitude equations (20) immediately tells us the *scaling* behavior with small ε of the *physical* length, time, and pattern intensity. For example, we expect the time dependence of Eq. (20) to occur on an $O(1)$ time scale with respect to the variable T. The relationship to the physical time scale then shows us that the physical time scale, for example for the growth of a small initial perturbation from the spatially uniform state, will be $\tau_0 \varepsilon^{-1}$ which diverges toward threshold as ε^{-1}, with ε the small parameter that measures the distance of the control parameter from threshold and goes to zero at threshold. Similarly, the length scale over which the intensity of the pattern grows from a suppressed value near a boundary or the core of a defect will be $\xi_0 \varepsilon^{-1/2}$ for the direction perpendicular to the stripes. It is in fact variations on these scales that are "slow" enough to be captured by the amplitude equation. Note that the length scale for the spatial variation corresponds in Fourier space to a wave number deviation $q - q_c = O(\varepsilon^{1/2} \xi_0^{-1})$. This is the order of the width of the unstable band near threshold, so that all of these states are accessible to the amplitude equation treatment. Finally, the amplitude

[5] A caveat we must make is that there are no degeneracies, for example other bifurcations occurring at the same parameter value which would lead to other amplitudes varying slowly in space and time that might couple to A, and no other slowly varying degrees of freedom.

of the pattern will have the characteristic square root dependence proportional to $\sqrt{\varepsilon}$, so that the intensity of the pattern proportional to $|A|^2$ will grow linearly in ε.

These power law dependencies are quite analogous to the divergences such as "critical slowing down" that occur at a second order phase transition, and indeed have the same parameter dependence as in the mean field approximate description of that phenomenon.

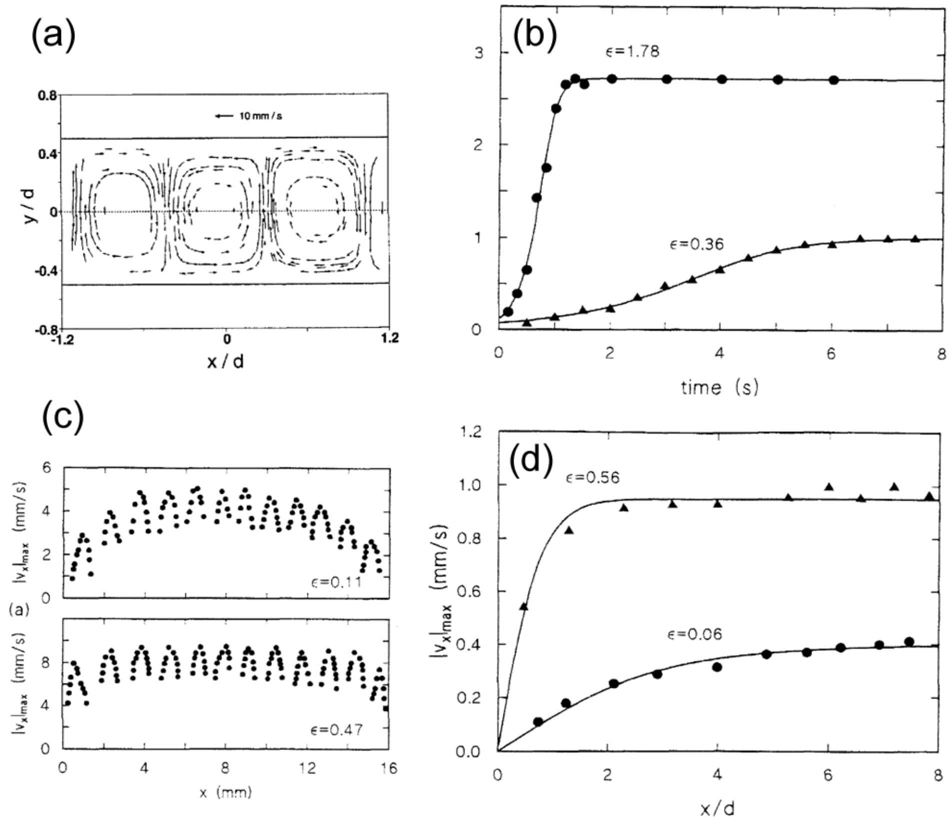

Figure 2: Smectic Films", by S. W. Morris, J. R. deBruyn, and A. D. May, J. Stat. Phys. **64**, 1025 (1991), and "Electroconvection Patterns in Smectic Films at and Above Onset " by S. S. Mao, J. R. deBruyn, and S. W. Morris, Physica A **239**, 189 (1997)]

There have been numerous direct experimental verifications of these scaling results. As an example, the results for an experiment on convection in a thin liquid film are shown in Fig. 2. The liquid in these experiments was actually a smectic liquid crystal which has a layered structure that stabilizes the uniformity of the thickness of the film. The properties of the flow parallel to the film are, however, the same as in a conventional liquid. The convection is driven electrically, which couples to the fluid motion through charged ion impurities dissolved in the film. These experiments are particularly simple to interpret since the flow is accurately two dimensional, and so the pattern formation is a one-dimensional phenomenon (i.e. there is one confined and one extended dimension, so that the system is described by an amplitude equation in one space dimension).

The velocity of the convective flow was measured from the motion of tracer particles, as indicated in Fig. 2a, giving a quantitative measurement of the quantity that defines the pattern forming state, Fig. 2c. An envelope fitted to the measured velocity field yields the amplitude, and to a reasonable approximation the maxima of the velocity curve in panel (c) can be used to estimate the magnitude of the amplitude function at those points, so that we can interpret the velocity magnitudes plotted in panels (b) and (d) as the magnitude of the amplitude function. Panel (b) shows the time dependence of the maximum flow velocity (which corresponds to the amplitude away from the boundaries) from a small initial magnitude to saturation, and the increase near threshold of the time for this process consistent with the scaling of ε^{-1}. The spatial variation of the amplitude near the side boundary, where the flow velocity is suppressed, is shown in panel (d). The variation of the length scale over which the amplitude recovers is again consistent with the expected scaling with $\varepsilon^{-1/2}$. In both panels (b) and (d) the change of the saturated amplitude with ε can be read off from the large time or distance value. The experimentalists also verified quantitatively that this amplitude increases as $\varepsilon^{1/2}$.

Potential Dynamics

The analysis and understanding of the amplitude equation are vastly simplified by a remarkable feature of the equation, namely the existence of a **potential** or a **Lyapunov function**. This quantity is an integral functional of the amplitude and low order derivatives over the domain that for periodic, and a few other boundary conditions, has properties analogous to the mechanical energy (potential and kinetic) of a frictionally damped ball in a potential, namely that the functional monotonically decreases in the dynamics. Just as for the damped ball, for which we know that the motion will eventually cease with the ball at a minimum of the mechanical energy (zero kinetic energy and potential energy at a minimum, not necessarily the lowest), the potential for the amplitude equation tells us that the amplitude will approach a configuration giving a minimum of the potential, and that here the dynamics will cease. This is a very restrictive result that provides strong constraints on the dynamics. The existence of a potential often provides a powerful tool for understanding the system. We refer to the existence of the potential as remarkable, because it is not a property that we would generally expect for a system far from equilibrium. Indeed, more careful analysis shows that the existence of this type of potential is an artifact of the lowest order truncations in the expansions leading to the amplitude equation, since higher order amplitude equations are generally no longer potential.

Let us show that the dynamics predicted by the lowest order amplitude equation is potential for certain kinds of boundaries. The dynamics of the amplitude predicted by equation Eq. (7), together with particular but common boundary conditions, is easily shown to be consistent with the continual decrease of the potential V given by

$$V[A] = \iint dx \left[-\varepsilon |A|^2 + \frac{1}{2}|A|^4 + \xi_0^2 \, |\partial_x A|^2 \right]. \tag{21}$$

This quantity evolves according to

$$\frac{dV}{dt} = -2\tau_0 \int \int dx \ \ |\partial_t A|^2 \,, \tag{22}$$

which necessarily increases in any dynamics of A.

This can be verified as follows. Taking the time derivative of Eq. (21) yields

$$\frac{dV}{dt} = \iint dx \left\{ \left(-\varepsilon A + |A|^2 A\right) \frac{\partial A^*}{\partial t} + \xi_0^2 (\partial_x A) \left(\partial_x \frac{\partial A^*}{\partial t} \right) + \text{c.c.} \right\} \tag{23}$$

We now integrate by parts in the last term, to remove the spatial derivatives from $\partial A^*/\partial t$. This operation generates terms that are evaluated at the boundaries of the domain. For V to be a potential, these boundary terms must vanish. This is the case for periodic boundary conditions, and for the boundary conditions Eq. (17). If the boundary terms disappear we then find

$$\frac{dV}{dt} = \iint dx \left\{ \left[-\varepsilon A + |A|^2 A - \xi_0^2 \partial_x^2 A \right] \frac{\partial A^*}{\partial t} + \text{c.c.} \right\}. \tag{24}$$

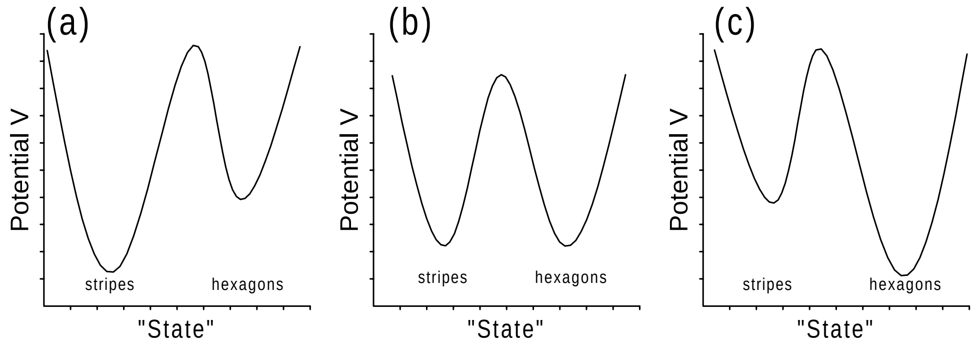

Figure 3: Nonlinear competition for a potential system. The variation of the potential between states of ideal stripes and hexagons is shown for the cases where the stripe state has the lower potential (panel a), the stripe and hexagon states have the same potential (panel b), and the hexagon state has the lower potential (panel c). In general there will be a potential maximum, or *barrier*, between the two minima, so that dynamics from the higher minimum to the lower cannot be inferred.

Observing that the term in the square braces is just $-\partial A/\partial t$ gives us Eq. (22).

Equation (22) tells us that if there is *any* dynamics of A the potential V decreases monotonically in time. A quantity that decreases monotonically in time is called a Lyapunov function, and dynamics given by a variational form such as Eq. (22), a stricter statement, is said to be *potential* or *relaxational*. This type of dynamics is also sometimes called *gradient flow*. Potential dynamics is strongly constrained. In particular if the potential is bounded from below, as is the case for Eq. (21), persistent dynamics on a time scale that does not increase indefinitely is ruled out— and there is no periodic, quasiperiodic or chaotic dynamics in such systems. Instead the dynamics runs "down hill" in V until the amplitude A reaches a minimum of the potential, when the dynamics will cease.

The existence of a potential leads to many useful deductions. An example is the question of the competition between two patterns (such as a stripe state and hexagonal state[6]), if both are present in the system with a wall or domain boundary between them. We can argue that if there is any motion of the wall, it must be in the direction that increases the fraction of the pattern with the lower value of the potential. (Note that there will be a contribution to the potential from the domain wall itself, but this does not change as the wall translates.) Thus the parameter value for which the two states have equal potentials can be used to identify the point at which the preferred pattern switches from one to the other. Two caveats should be stated. The first is that the result applies *only in the context of an experiment in which the competition between bulk saturated regions, in contact via a domain wall, occurs*. Other experimental conditions, such as the growth from small initial conditions, may give different results. This is because in general there is a potential barrier between the two ideal states, see Fig. 3, and only in special physical circumstances is there a dynamical path between the two that flows monotonically down the potential. Secondly, the motion may be impeded, for example by pinning of the wall to the stripes themselves, in which case there may be no motion even if the potentials are different. This would then give a finite range of parameters for coexistence.

A second application of the potential is to the question of wave number selection, the precise value of the wave number in a stripe state (or unit cell size in the lattice states). Again we can argue that any local dynamics that mediates between two ideal states occupying large portions of the system, which will then dominate the integral that forms the potential, will favor the state with a lower value of the potential $V(q)$. (Here $V(q)$ is

[6]The amplitude equations for a state of superimposed stripes remains potential.

the potential evaluated for the ideal periodic state with wave number q.) In the case of potential dynamics, different dynamical mechanisms that allow the wave number to change, such as dislocation motion, boundary relaxation etc., will all tend to yield the *same* wave number, namely the one that minimizes $V(q)$. For systems without a potential, there is no such argument, and different dynamical mechanisms may lead to different wave numbers.

Applications of the Amplitude Equation

Lateral Boundaries

An early vexing question in understanding pattern formation was the degree to which the ideal states of theory, based on laterally infinite systems or systems with periodic boundary conditions, had anything to do with the states seen in experiments on necessarily finite systems. Further there was question of how the properties of a laterally "large" system approached those of the infinite system. For systems that are large compared with the pattern periodicity, and for control parameter values close to onset, the amplitude equation gives us a formalism that can readily address these issues. The approach is particularly well suited to systems with one extended coordinate x, or systems with two extended directions with a pattern of stripes parallel to the boundary, since in these situations the formalism simplifies to a one dimensional amplitude equation. The general situation in a two dimensional system is harder, since the boundaries often tend to reorient the stripes, leading to a pattern with large reorientations of stripes, that cannot be treated within the amplitude equation description.

We will here study the case of boundaries that tend to inhibit the pattern formation. For steady states with a one dimensional spatial variation and suppressing boundaries we must solve the amplitude equation (written in the fully scaled form) Eq. (20) with no time variation

$$0 = \bar{A} + \partial_X^2 \bar{A} - |\bar{A}|^2 \bar{A}. \tag{25}$$

This must be solved with the condition at the boundaries corresponding to Eq. (17a)

$$\bar{A} = 0. \tag{26}$$

The amplitude equation then allows us to determine how the intensity of the pattern grows with distance away from the boundary, to approach the bulk saturated value far away. There are also dramatic effects on the range of possible wave numbers of the pattern far away from the boundary, expressed through restrictions on the phase variation of the complex amplitude. We will see that the stationary solutions to Eqs. (25,26) in fact have a *constant* phase, so that, within the accuracy of the lowest order amplitude equations, the wave number of the stripes is precisely the critical wave number. Compare this with the infinite or periodic system, where there is a wide band of stable, stationary solutions of different wave numbers limited only by stability considerations.

We illustrate how the amplitude equation can be used to understand the effect of lateral boundaries by considering first the case of a semi-infinite one dimensional system $X > 0$, with a suppressing boundary at $X = 0$. It can be verified by substitution that a solution to the amplitude equation Eq. (25) and the boundary condition Eq. (26) at $X = 0$ is

$$\bar{A} = e^{i\phi} \tanh(X/\sqrt{2}), \tag{27}$$

where ϕ is a real constant. This becomes in the unscaled variables the expression

$$A = e^{i\phi} \sqrt{\frac{\varepsilon}{g_0}} \tanh(\frac{x}{\xi}), \tag{28}$$

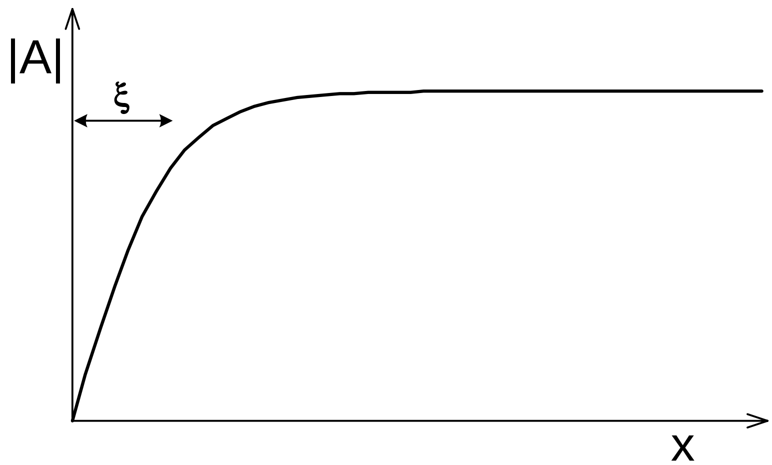

Figure 4: Plot of the amplitude as a function of position x near a boundary at $x = 0$ where the amplitude $A = 0$. The arrow shows the length of the healing length $\xi = \sqrt{2}\xi_0\varepsilon^{-1/2}$, which sets the length scale for the variation of the magnitude.

with

$$\xi = \sqrt{2}\xi_0\varepsilon^{-1/2}. \tag{29}$$

The form of the magnitude $|A|$ of the solution is shown in Fig. 4. This simple solution demonstrates many of the important features of the effect of boundaries in suppressing the pattern nearby. We see that the suppression of the amplitude from its bulk saturated value extends over a *healing length* or *coherence length* ξ that diverges as $\varepsilon^{-1/2}$ towards threshold. This is the characteristic length scale for variations of the magnitude $|A|$. The result that the suppression extends over many stripe widths near threshold is quite surprising without the insights of the amplitude equation approach. This prediction is amply confirmed by experiment: Eqs. (28) and (29) were used in constructing Fig. 2d, and Fig. 4 directly corresponds to that figure.

The solution Eq. (28) contains an arbitrary constant phase factor, corresponding physically to any positioning of the stripes relative to the boundary. There are no solutions with a spatially varying phase, which would correspond to a deviation of the wave vector of the stripes from the critical wave vector. Thus, far away from the boundary where the magnitude has saturated, we have a stripe state with wave number completely determined at $O(\varepsilon^{1/2})$

$$q = q_c + 0 \times \varepsilon^{1/2} + O(\varepsilon). \tag{30}$$

Compare this to the laterally infinite or periodic system for which states exist over a wave number band that grows as $\varepsilon^{1/2}$ above threshold. These conclusions are modified when the calculation is extended to higher order in ε. In those more extended calculations it is found that the phase does vary in space, but in a manner consistent with Eq. (30). The solution far away from the side wall again tends towards saturated stripes, now with a wave number somewhere in a narrow band, which has a width that scales linearly with ε near threshold, rather than as $\varepsilon^{1/2}$ as in the periodic or infinite system. In addition, the stripe positions relative to the boundary become restricted to a discrete set of values.

Now consider a finite geometry $0 \le X \le L$ with boundary conditions $\bar{A}(0) = \bar{A}(L) = 0$. For large L the regions of suppressed magnitude near the boundaries are far apart, and can be treated independently, so that $|\bar{A}|$ has a "top hat" type X dependence, saturating at $\bar{A} = 1$ in the bulk away from the boundaries, as shown by the solid curve in Fig. 5. As L is reduced, the suppression regions begin to overlap, and the

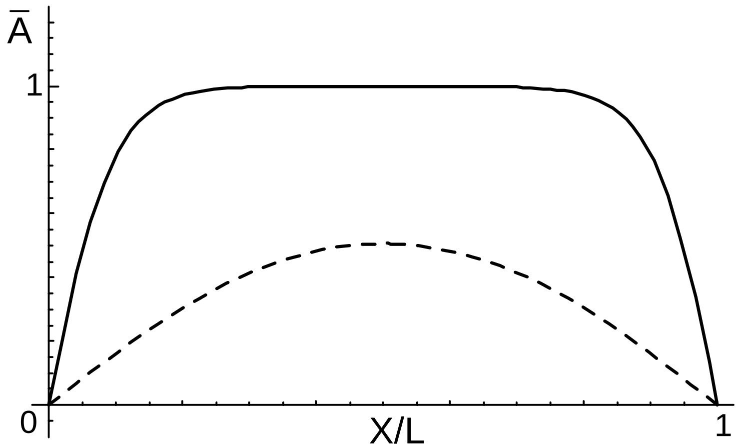

Figure 5: Solution of the fully scaled amplitude equation Eq. (25) in a finite geometry of size L with boundary conditions $\bar{A} = 0$ at $X = 0, L$, plotted as a function of X/L. The full curve is for $L = 15$ (physical size $l = 15\varepsilon^{-1/2}\xi_0$), and the dashed curve is for $L = 3.5$ (physical size $l = 3.5\varepsilon^{-1/2}\xi_0$). Note that for system sizes large compared to the healing length, the amplitude away from the boundaries saturates at the bulk saturated value, whereas for sizes comparable to the healing length, the amplitude does not reach this value. For $L < \pi$ (corresponding to $\varepsilon < \pi^2\xi_0^2/l^2$ in unscaled units) there is no nonzero solution. For L slightly larger than π, the solution is proportional to $\sin \pi X/L$, which is the solution to the linearized amplitude equation.

maximum amplitude is reduced below the bulk saturation, as for the dashed curve in Fig. 5. For smaller L, the maximum amplitude decreases, and we may eventually use a linear approximation to the amplitude equation This yields the linear onset solution in the finite geometry

$$\bar{A} = \bar{a}e^{i\phi} \sin X. \tag{31}$$

The magnitude prefactor $\bar{a}$ is not determined by the linear equation[7], but the solution only satisfies the boundary conditions if $L = n\pi$, $n = 1, 2, \ldots$. Translating to the unscaled units, in which the system size is l with $L = \varepsilon^{1/2} l/\xi_0$, we see that the onset (the $n = 1$ solution) occurs at the shifted value of the control parameter

$$\varepsilon_c = \pi^2 \xi_0^2 / l^2. \tag{32}$$

This is an explicit calculation of the suppression of the onset by finite size effects in the case of suppressing boundaries. The solution Eq. (31) again contains an arbitrary constant phase factor to a continuum of onset solutions with different stripe positions. If the amplitude equation is extended to higher order, this degeneracy is removed, to give a discrete set of onset solutions with values of ε_c as in Eq. (32), together with small corrections of order $(\xi_0/l)^4$.

Eckhaus Instability

The amplitude equation provides a direct way to investigate the instability of ideal states with respect to spatially dependent perturbations, and so to construct the stability balloon near onset. The universal form of

[7] The magnitude $\bar{a}$ can be determined by a Galerkin type approach, substituting this form back into the *nonlinear* amplitude equation, and then collecting terms in $\sin X$, whilst ignoring the higher harmonic term in $\sin 3X$ that is developed by the nonlinearity.

the equation implies that the stability balloon too will have universal features near onset. In addition we can learn much more about the instabilities, for example how the wave vector of the fastest growing perturbation varies as we move into the unstable region. Further, numerical simulations of the amplitude equation can be used to follow the growth of the perturbation to large amplitudes, so that the wave number changing dynamics, for example involving the elimination or creation of stripes, can be followed to completion.

The scheme of attack for a linear stability analysis is the standard one: first construct the unperturbed solution (here the nonlinear saturated steady state with a wave vector deviating form critical), and then investigate the dynamics of small perturbations linearizing about the base state. With the one dimensional amplitude equation we can study the stability of stripe states to longitudinal perturbations.

The stability balloon is given by testing the stability of states as a function of their wave number. The stripe state with wave vector differing slightly from q_c is given by the amplitude (in the scaled representation)

$$\bar{A}_K(X) = a_K e^{iKX}, \tag{33}$$

where the phase factor gives the wave number shift of the stripes

$$q = q_c + \xi_0^{-1}\varepsilon^{1/2}K, \tag{34}$$

and the magnitude prefactor is obtained as a simple result of substitution into the amplitude equation Eq. (20)

$$a_K^2 = 1 - K^2. \tag{35}$$

The existence band

$$-1 \le K \le 1, \qquad q_c - \xi_0^{-1}\varepsilon^{1/2} \le q \le q_c + \xi_0^{-1}\varepsilon^{1/2} \tag{36}$$

is the width of the band of wave numbers between the neutrally stable wave numbers.

The stability of these states is tested by adding to $\bar{A}_K$ a small perturbation $\delta\bar{A}$

$$\bar{A}(X,Y,T) = \bar{A}_K(X) + \delta\bar{A}(X,T). \tag{37}$$

By linearizing the amplitude equation, we see that the perturbation evolves according to the following linear evolution equation:

$$\partial_T \delta\bar{A} = \delta\bar{A} + {\partial_X}^2\delta\bar{A} - 2|\bar{A}_K|^2\delta\bar{A} - \bar{A}_K^2\delta\bar{A}^*. \tag{38}$$

The solution to Eq. (38) for $\delta\bar{A}$ turns out to be messy because of the spatial dependence of the coefficient $\bar{A}_K^2$ of the last term. Fortunately, since we are looking at the perturbation to a spatially periodic state, a version of Bloch's theorem applies, so that the stability eigenvalues and eigenvectors can be labelled by a Bloch wave vector Q (we will see that the perturbation $\delta\bar{A}$ actually has components with wave vectors $K \pm Q$). The task is then to calculate the exponential growth rate $\sigma_K(Q)$, which will depend on both the wave number K of the base state, and the Bloch wave vector Q characterizing the perturbation.

The usual form of Bloch's theorem addresses the properties of a perturbation to a real solution. To study Eq. (38) we need to generalize Bloch's theorem for a complex base state. The form of the generalization is shown by trying the ansatz for the perturbation in the form $\delta\bar{A} \sim e^{iKX}e^{iQX}$. Substitution into Eq. (38) gives a number of terms with the same spatial dependence, but also generates a term $e^{iKX}e^{-iQX}$. Thus we make the more general ansatz

$$\delta\bar{A} = e^{iKX}[\delta a_+(t)e^{iQX} + \delta a_-^*(t)e^{-iQX}] \tag{39}$$

(where we use the complex conjugate on δa_-^* for later convenience). Now substituting into Eq. (38), linearizing in $\delta a_\pm$, and collecting the coefficient of the two independent functions $e^{i(KX\pm QX)}$ gives the pair of equations

$$d_t\delta a_+ = -(P^2 + U_+)\delta a_+ - P^2\delta a_-, \tag{40a}$$

$$d_t\delta a_- = -P^2\delta a_+ - (P^2 + U_-)\delta a_-, \tag{40b}$$

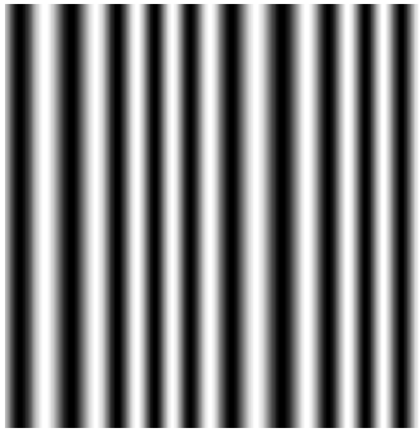

Figure 6: Sketch of a stripe system undergoing an Eckhaus instability which is a modulation of the wave length of the stripes. The onset of instability occurs with a length scale of the modulation that is large compared with the stripe width.

with

$$P^2 = 1 - K^2, \tag{41}$$

and

$$U_\pm = [K \pm Q_X]^2 - K^2. \tag{42}$$

The growth rate $\sigma_K(Q)$, defined by $\delta a_\pm \sim \exp[\sigma_K(Q)t]$ is given by a standard matrix eigenvalue calculation. The algebra eventually leads to the following expression for the most positive growth rate

$$\sigma_K(Q) = -P^2 - \frac{1}{2}(U_+ + U_-) + [P^4 + \frac{1}{4}(U_+ - U_-)^2]^{1/2}. \tag{43}$$

This is the growth rate function that tells us the stability of the state at wave number K.

We test the stability of a base state solution with wave number K, by finding the maximum growth rate $\sigma_K(Q)$, Eq. (43), as a function of the perturbation wave vector Q. The state K is stable if this maximum growth rate is negative. It turns out that for the form of $\sigma_K(Q)$ in Eq. (43), as K is increased from zero (q moves away from q_c) where the base state is stable, the instability always occurs first for a long wavelength disturbance, i.e. in the limit $Q \to 0$.

For a perturbation characterized by the Bloch wave vector Q, the growth rate is

$$\sigma_K(Q) = (1 - K^2) - Q^2 + \sqrt{(1 - K^2)^2 + 4K^2Q^2}. \tag{44}$$

This is a type II instability, as can be shown by expanding in small Q:

$$\sigma_K(Q \to 0) = -\left(\frac{1 - 3K^2}{1 - K^2}\right) Q^2 - \left(\frac{2K^4}{(1 - K^2)^3}\right) Q^4 + O(Q^6). \tag{45}$$

Since the coefficient of the Q^4 term is always negative within the existence band $|K| < 1$, we see that the stability boundary occurs when the coefficient of the Q^2 term becomes negative, and the most unstable mode is a long wavelength perturbation, $Q \to 0$. The instability occurs for

$$|K| > \frac{1}{\sqrt{3}}. \tag{46}$$

Thus the band of wave numbers near threshold that are stable to the longitudinal insatiably is $1/\sqrt{3}$ times the width of the existence band, independent of any details of the system. Returning to physical units, the longitudinal instability occurs at $q = q_c \pm k_E$ with

$$k_E = \frac{1}{\sqrt{3}} \xi_0^{-1} \varepsilon^{1/2}, \tag{47}$$

and the width of the stable band of wave numbers $2k_E$ grows as the square root of the distance above onset.

This instability is named after Walter Eckhaus, who first studied the instability in 1965. The form of the perturbation is a sinusoidal spatial modulation of the wave number of the pattern, with regions of compression and stretching, Fig. 6. The result for the boundary of the stability balloon, Eq. (46), can be obtained by a simpler calculation using the phase equation, §7. The present calculation gives us additional insights into the instability. For example, it can be shown from Eq. (44) (see Exercise **??**) that for wave numbers K unstable to the Eckhaus instability, the maximum growth rate occurs for a perturbation of wave number

$$Q_{\max}^2(K) = 3\frac{(K^2+1)(3K^2-1)}{4K^2}, \tag{48}$$

for which the growth rate is

$$\sigma_{\max}(K) = \frac{(3K^2-1)^2}{4K^2}. \tag{49}$$

These results for the instabilities teach us important general lessons. We see that the instability boundary for the Eckhaus instability takes on a universal quantitative form near threshold. Since the stability balloon gives us our basic understanding of the periodicities available for pattern formation, this is an important insight. As in the analysis about the uniform state, the linear stability analysis leaves us with an exponentially growing perturbation in the unstable regions. We need to study effects nonlinear in the perturbation of the stripe state to understand the subsequent fate of the stripes, such as the important question of whether the perturbation saturates at small amplitude, or continues to catastrophically change the pattern, for example by eliminating a stripe pair in the Eckhaus instability). This cannot be done analytically, but it is quite easy to simulate the amplitude equations numerically.

Phase Dynamics

The magnitude a and phase ϕ of the complex amplitude play different dynamical roles in the description of pattern formation. In particular, a perturbation of a will tend to relax to the value determined by the nonlinear terms in the amplitude equation on time scale $\varepsilon^{-1}\tau_0$. On the other hand, a phase perturbation that is independent of position does not relax at all, and the relaxation of a perturbation on a length scale l would be expected to relax on a time scale that diverges with l (as l^2 or longer, as we will see). We can therefore imagine situations where the phase is relaxing much more slowly than the magnitude, so that the magnitude can be evaluated as the value consistent with the instantaneous phase field as if this were time independent. Technically, we neglect terms in $\partial a/\partial t$, since these are small compared with other terms in the dynamical equation for the magnitude, such as εa. The magnitude is said to **adiabatically** follow the phase variation, and the approximation method is known as **adiabatic elimination**. This approach allows us to derive a simple dynamical equation for the slow phase variation, known as the *phase equation*. Since a change in the phase at some position corresponds to a displacement of the pattern at that point, the phase equation captures some of the essential features of pattern dynamics.

For simplicity of notation, we again consider the amplitude equation in its scaled form Eq. (16). Our goal is to find dynamical equations for the slow variation of ϕ due to long wave length perturbations. Although the calculation can be done more generally, we choose to look at small perturbations from a uniform stripe state at a wave number q shifted from critical $q = q_c + \varepsilon^{1/2} K \xi_0^{-1}$ that is described by the time independent amplitude as in Eq. (33)

$$\bar{A} = a_K e^{iKX}, \tag{50}$$

with the value of the constant $a_K = \sqrt{1-K^2}$ as in Eq. (35). We now write for the perturbed amplitude

$$\bar{A}(X,T) = a e^{i\phi} e^{iKX}, \tag{51}$$

with $a = a_K + \delta a(X,T)$, and linearize in small spatial derivatives of the phase $\phi(X,T)$ and small amplitude perturbations $\delta a(X,T)$. Since the length scale of the perturbations is supposed long, we neglect higher-order spatial derivatives of the same quantity, e.g. $\partial_X^2 a \ll \partial_X a$, and we only keep terms that lead to terms in the final equation for the phase up to derivatives of ϕ that are second order.

The formal scheme is to insert Eq. (51) into the amplitude equation Eq. (16), multiply through by $e^{-iKX}e^{-i\phi}$ and collect real and imaginary parts. Using

$$\partial_T A = (\partial_T a + i a \partial_T \phi) e^{iKX} e^{i\phi}, \tag{52}$$

shows, after multiplying through by $e^{-iKX}e^{-i\phi}$, that the real part of the equation will give the dynamical equation for a, and the imaginary part the dynamical equation for ϕ.

Keeping only terms linear in δa and ϕ and up to second order derivatives, the real part gives

$$\partial_T \delta a = -2a_K^2 \delta a - 2K a_K \partial_X \phi + \partial_X^2 \delta a. \tag{53}$$

Note that for a spatially uniform perturbation, Eq. (53) shows that the magnitude perturbation δa relaxes exponentially as $\exp(-a_K^2 T)$. Since a_K^2 is of order unity, this is a rapid decay of magnitude perturbation as discussed in the introduction to this section. The phase variations drive a nonzero value of δa. In comparing the size of the terms in Eq. (53) involving δa we see that the dominant term is the first one on the right hand side, since all the other terms involve spatial derivatives or time derivatives of δa that are small for slow variations. Thus

$$a_K \delta a \simeq -K \partial_X \phi, \tag{54}$$

and δa adiabatically follows the perturbations of the phase gradient. Note that this is just the equation for the change in magnitude given by Eq. (35) arising from a change in the wave number K by $\partial_X \phi$.

The imaginary part of Eq. (52) multiplied by $e^{-iKX}e^{-i\phi}$, and again keeping only terms linear in δa and ϕ and up to second order derivatives, gives

$$a_K \partial_T \phi \simeq 2K \partial_X \delta a + a_K \partial_X^2 \phi. \tag{55}$$

Eliminating δa using Eq. (54) and using Eq. (35) leads to the following evolution equation for small phase perturbations

$$\partial_T \phi = \left[\frac{1-3K^2}{1-K^2}\right] \partial_X^2 \phi. \tag{56}$$

This is a *diffusion equation* for the phase, with diffusion constant $D_\parallel$ for variations along the stripe normal. Transforming back to the unscaled space and time variables, the equation becomes

$$\partial_t \phi = D_\parallel \partial_x^2 \phi, \tag{57}$$

with diffusion constant for phase perturbations about the stripe state with wave number $q = q_c + k$ (with $k = \xi_0^{-1}\varepsilon^{1/2} K$)

$$D_\parallel = (\xi_0^2 \tau_0^{-1}) \frac{\varepsilon - 3\xi_0^2 k^2}{\varepsilon - \xi_0^2 k^2}. \tag{58}$$

The phase dynamics is the Goldstone mode that reflects the broken translational symmetry of the stripe state. Consequently, the phase equation is a powerful tool, and many important results can be derived from it. For example, we know that diffusion equations lead to exponentially growing solutions if the diffusion constant is negative, signalling instability. We therefore see that the state with wave number $q_c + k$ is unstable to long wavelength longitudinal phase perturbations for $|\xi_0 k| > \varepsilon^{1/2}/\sqrt{3}$. This is just the Eckhaus instability described previously in Section 7.

Although the phase dynamics is easily derived from the amplitude equation formalism as we have just done, it has a wider validity. Even away from threshold, the symmetry aspects of the pattern are captured by an appropriately defined phase variable. Again slow spatial variations of the phase necessarily evolve slowly in time, and this slow variation can be isolated mathematically from the faster dynamics of other degrees of freedom in the form of a "phase equation". The phase dynamics provides a simple way to investigate some important questions such as what are some of the instabilities that bound the stability balloon of the finite-amplitude nonlinear stripe states. It can also be used to study patterns that change their orientation over large angles.

Limitations of the Amplitude Formalism

Although many interesting questions can be addressed within the amplitude equation, it is important to bear in mind the limitations of the formalism.

The amplitude equation is derived by perturbation expansion and truncation, and so is only a good approximation over a restricted range of parameters, in particular near onset, and for long wave length and temporally slow modulations of the ideal pattern.

There are limitations on the nature of the patterns that can be treated. For example, because of the lack of a rotationally invariant formulation, the only patterns that can be calculated quantitatively are those that are close to a single set of parallel stripes, or, in the two dimensional generalization to be studied in the next chapter, a superposition of stripes such as squares and hexagons. Patterns in which the orientation of stripes or lattices vary through large angles over large distances cannot be treated even though the rate of variation may be slow.

The way in which the approximation is "good" may be quite subtle. Indeed the answers to qualitative question may be quite wrong! For example if we ask the question "Can system ABC show chaos near onset?" the amplitude equation immediately leads us to the answer "No", because of the existence of the potential. However, since the equation is derived as an approximation, we should not be so definite in any physical statement. Indeed, the correct answer might be: "The relaxational dynamics predicted by the amplitude equation should be a good gross description of what happens." However at very long times, there may be very slow persistent dynamics at a time scale beyond the $O(\varepsilon^{-1})$ time scale of the dynamics controlled by the amplitude equation, or there may be small amplitude persistent dynamics, perhaps on a fast scale, that is again outside of the control of the amplitude equation. Alternatively, the amplitude equation may predict dynamics that is quenched by residual effects not captured by the perturbation formalism. We have said that the amplitude equation is good "near onset". We might also question more carefully the this phrase: does this mean asymptotically near onset, or just at some small but finite distance from onset? Finally, as a worst case, if the solutions predicted by the amplitude equation are actually very sensitive to slight changes in the equations (such solutions are called structurally unstable), the predicted behavior might be quite misleading and bear no resemblance to the actual physical behavior.